Influence Indices

Michel Grabisch, Agnieszka Rusinowska

GATE
Groupe d'Analyse et de Théorie Économique
UMR 5824 du CNRS

DOCUMENTS DE TRAVAIL - WORKING PAPERS

W.P. 07-05

Influence Indices

Michel Grabisch, Agnieszka Rusinowska

Mars 2007

GATE Groupe d'Analyse et de Théorie Économique
UMR 5824 du CNRS
93 chemin des Mouilles – 69130 Écully – France
B.P. 167 – 69131 Écully Cedex
Tél. +33 (0)4 72 86 60 60 – Fax +33 (0)4 72 86 60 90
Messagerie électronique gate@gate.cnrs.fr
Serveur Web : www.gate.cnrs.fr

Influence Indices*

MICHEL GRABISCH[1] and AGNIESZKA RUSINOWSKA[2]

[1] Université Paris I Panthéon-Sorbonne
CERMSEM, 106-112 Bd de l'Hôpital, 75013 Paris, France
Michel.Grabisch@lip6.fr
[2] GATE, CNRS UMR 5824 - Université Lumière Lyon 2 - Ecole Normale Supérieure LSH
93, Chemin des Mouilles - B.P.167, 69131 Ecully Cedex, France
rusinowska@gate.cnrs.fr

Abstract. In the paper, we investigate the Hoede-Bakker index - the notion which computes the overall decisional 'power' of a player in a social network. It is assumed that each player has an inclination (original decision) to say 'yes' or 'no' which, due to influence of other players, may be different from the final decision of the player. The main drawback of the Hoede-Bakker index is that it hides the actual role of the influence function, analyzing only the final decision in terms of success and failure. In this paper, we further investigate the Hoede-Bakker index, proposing an improvement which fully takes into account the mutual influence among players. A global index which distinguishes an influence degree from a 'power' index is analyzed. We define weighted influence indices, in particular, a possibility influence index which takes into account any possibility of influence, and a certainty influence index which expresses certainty of influence. We consider different influence functions and study their properties.

Keywords: Hoede-Bakker index, weighted influence index, possibility influence index, certainty influence index, equidistributed influence index, influence function

JEL Classification: C7, D7

1 Introduction

In cooperative game theory, a *decisional power* has been proposed by Hoede and Bakker [10], and later generalized and modified by Rusinowska and de Swart [19]. One may ask a question why it is interesting to analyze this concept, in particular, since a lot of indices have been studied in the voting power literature (see, for instance, [1], [3] - [6], [11], [13], [14], [16], [17], [20], also [8] for an overview). Generally speaking, the Hoede-Bakker index computes the overall decisional 'power' of a player in a social network. It is assumed that a decision of a player may be influenced by decisions of other players. Specifically, it is considered that each player has an *inclination* (original decision) to say 'yes' (coded by $+1$) or 'no' (coded by -1). For each possible configuration i of individual inclinations, it is supposed that after mutual influence the actual decision Bi of all players is made. Then, a group decision $gd(Bi)$ is given.

The main drawback of the Hoede-Bakker index is that it hides the actual role of the influence function B, analyzing only the final decision in terms of successes and failures. The aim is then to provide alternative ways putting into lights the role of the influence function B.

In the paper, we propose a general form of the index which enables the analysis of influence among players. This index fills a gap between power indices which are classical

* This research has been initiated during a Short Term Scientific Mission of Michel Grabisch to Tilburg University. The authors gratefully acknowledge support of this mission by COST Action 274, TARSKI.

in voting games (e.g. the Banzhaf index), and the Hoede-Bakker index. The general idea is to compute a weighted number of times an individual j makes another individual k change his decision, and more generally, the number od times a group S makes an individual $k \notin S$ change his decision. Two particular ways of weighting lead to a possibility influence index which takes into account any possibility of influence, and to a certainty influence index which expresses certainty of influence.

When analyzing an influence among players, we come naturally to a concept of followers, where by a follower of a given player we mean the voter who always decides according to the inclination of the player in question. In the paper, we study some properties of the set of followers of a given coalition of players.

By proposing different influence functions B and the group decisions gd, we usually receive different Hoede-Bakker indices. In this paper, we propose several influence functions (like, in particular, a majority function, a 'guru function', the identity function, and some others), and study their properties. We consider a purely influential function of one coalition upon another coalition, and a particular case of such a function, called the canonical pure influential function. Another related concept we study in this paper is the kernel of an influence function, that is, the set of 'true' influential coalitions.

2 The Hoede-Bakker index

Let us first recapitulate the original Hoede-Bakker index as introduced in [10], and its generalization given in [19]. The general framework is the following. We consider a social network with the set of all players (agents, actors, voters) denoted by $N = \{1, ..., n\}$. The players have to make a certain acceptance-rejection decision. Each player has an inclination either to say 'yes' (denoted by $+1$) or 'no' (denoted by -1). An *inclination vector*, denoted by i, is an n-vector consisting of ones and minus ones. Let I be the set of all inclination vectors. It is assumed that players may influence each others, and due to the influences in the network, the final decision of a player may be different from his original inclination. In other words, each inclination vector $i \in I$ is transformed into a *decision vector* Bi, where $B : I \to I$ is the influence function. The set of all influence functions will be denoted by $\mathcal{B}$. The decision vector Bi is an n-vector consisting of ones and minus ones and indicating the decisions made by all players. Let $B(I)$ denote the set of all decision vectors. Furthermore, the *group decision function* $gd : B(I) \to \{+1, -1\}$ is introduced, having the value $+1$ if the group decision is 'yes', and the value -1 if the group decision is 'no'. The set of all group decision functions will be denoted by $\mathcal{G}$.

Hoede and Bakker (1982) adopted the following two axioms to be satisfied by functions $B \in \mathcal{B}$ and $gd \in \mathcal{G}$:

AXIOM (GB-0):

$$\forall i \in I \; [gd(B(-i)) = -gd(Bi)] \tag{1}$$

AXIOM (GB-1):

$$\forall i, i' \in I \; [i \leq i' \; \Rightarrow \; gd(Bi) \leq gd(Bi')] \tag{2}$$

where

$$i \leq i' \iff \{k \in N \mid i_k = +1\} \subseteq \{k \in N \mid i'_k = +1\} \tag{3}$$

and $i < i'$ means: $i \leq i'$ and $i \neq i'$.

Assuming both axioms (GB-0) and (GB-1) to be satisfied, the following definition has been introduced ([10]):

Definition 1 *Given $B \in \mathcal{B}$ and $gd \in \mathcal{G}$, the decisional power index (the Hoede-Bakker index) $HB : \mathcal{B} \times \mathcal{G} \to [0,1]^n$ is given by*

$$HB_k(B, gd) := \frac{1}{2^{n-1}} \sum_{\{i | i_k = +1\}} gd(Bi) \quad for\ k \in N. \tag{4}$$

In [19], a certain generalization of the Hoede-Bakker index has been proposed. Let us notice that axiom (GB-0) may be easily violated, for instance, when there is a vetoer, or when the ability to influence depends on the inclination of the influencing player; see [19]. In order to make the Hoede-Bakker index more applicable, one may keep axiom (GB-1), and replace (GB-0) by two axioms (GB-2) and (GB-3) ([19]), which after adopting the notation:

$$1_N = (+1, ..., +1), \quad -1_N = (-1, ..., -1)$$

look as follows:

AXIOM (GB-2):

$$gd(B1_N) = +1 \tag{5}$$

AXIOM (GB-3):

$$gd(B(-1_N)) = -1. \tag{6}$$

Assuming all axioms (GB-1), (GB-2), and (GB-3) to be satisfied, the following definition has been introduced ([19]):

Definition 2 *Given $B \in \mathcal{B}$ and $gd \in \mathcal{G}$, the generalized Hoede-Bakker index GHB : $\mathcal{B} \times \mathcal{G} \to [0,1]^n$ is given by*

$$GHB_k(B, gd) := \frac{1}{2^n} \left(\sum_{\{i | i_k = +1\}} gd(Bi) - \sum_{\{i | i_k = -1\}} gd(Bi) \right) \quad for\ k \in N. \tag{7}$$

Of course, the axioms (GB-0) and (GB-1) imply the axioms (GB-1), (GB-2), and (GB-3).

First of all, we notice that neither in the original definition of the Hoede-Bakker index nor in its generalization mentioned above, the functions B and gd are considered separately. When calculating the (original or generalized) Hoede-Bakker index, only the relation between an inclination vector i and the group decision $gd(Bi)$ is taken into account. If we do not separate the two functions B and gd, we may define *Success*, *Failure* and *Decisiveness* of a player starting not from the final decision of the player in question, but from his inclination (for an analysis of success and decisiveness of a player in voting situations, see for instance [15]). Consequently, we may say that a player is *successful* if his inclination coincides with the group decision. Adopting such a definition of being successful, if all inclination vectors are equally probable, then the generalized Hoede-Bakker index is a kind of a 'net' Success (see [18]), i.e., it is equal to 'Success − Failure = Decisiveness', where Success, Decisiveness, and Failure of a player are defined as a probability that the player is successful, is decisive, and fails, respectively. Moreover, under such a definition of Success, if all inclination vectors are equally probable, then the generalized Hoede-Bakker index coincides with the absolute Banzhaf index; see [19].

3 The influence indices

In order to take fully into account the mutual influence among players and to separate the functions B and gd in the Hoede-Bakker setting, we introduce

$$\textit{global index} = (d, \phi) \tag{8}$$

where d determines the influence degree and ϕ is the revised Hoede-Bakker index. We will investigate these two components separately. In particular, we will impose separate axioms on the 'influence part' and the 'power part' of the global index.

It seems reasonable to assume that if all players have the same inclination, their decisions will coincide with their inclinations. Consequently, we adopt the following two axioms on $B \in \mathcal{B}$:

AXIOM (B-1):

$$B1_N = 1_N. \tag{9}$$

AXIOM (B-2):

$$B(-1_N) = -1_N. \tag{10}$$

When analyzing the 'influence part', the first question may appear how to measure a degree of influence of a player (or a coalition) on the other voters. The answer is not necessarily that straightforward if we can just observe the inclinations and the final decisions of the players in a multi-player social network. Suppose the final decision of player A is different from his inclination, but this decision coincides with the inclinations of two other players in the network, say, agents B and C. Was voter A's decision different from his inclination because of the unique influence of player B, or the unique influence of player C, or maybe A voted like this only because he faced an influence of the strong two-party coalition? These are the questions that not always can be answered univocally if apart from knowing the function B, we are not able to observe a 'real act of influencing among players'. Consequently, we introduce a family of *influence indices*.

3.1 The possibility influence index

Let us first introduce some notations. Let for $k, j \in N$

$$I_{k\to j} := \{i \in I \mid i_j = -i_k\} \tag{11}$$

$$I^*_{k\to j}(B) := \{i \in I_{k\to j} \mid (Bi)_j = i_k\} = \{i \in I \mid (Bi)_j = i_k = -i_j\}. \tag{12}$$

$I_{k\to j}$ and $I^*_{k\to j}(B)$ denote the set of all inclination vectors of *potential influence* of player k on player j, and the set of all inclination vectors of *observed influence* of k on j under given $B \in \mathcal{B}$, respectively. Of course, $I_{k\to k} = I^*_{k\to k}(B) = \emptyset$ for each $k \in N$ and $B \in \mathcal{B}$, and $|I_{k\to j}| = 2^{n-1}$.

Definition 3 *Given $B \in \mathcal{B}$, for each $k, j \in N$, the possibility influence index of player k on player j is defined by*

$$\overline{d}(B, k \to j) := \frac{|I^*_{k\to j}(B)|}{|I_{k\to j}|} = \frac{|\{i \in I \mid (Bi)_j = i_k = -i_j\}|}{2^{n-1}}. \tag{13}$$

$\overline{d}(B, k \to j)$ measures a degree of influence player k has on player j, taking into account any possibility of influence. Of course, for each $B \in \mathcal{B}$ and $k, j \in N$

$$\overline{d}(B, k \to j) \in [0,1], \quad \overline{d}(B, k \to k) = 0.$$

In definition (13) we do not impose an additional condition that $(Bi)_k = i_k$, by which we allow the influencing player k to be in the same time influenced by another player. Although voter k is able to make player j change his inclination (preliminary decision), in the meantime player k may change his own preliminary decision as well.

Generalizing, we will also consider an influence of a coalition. Concerning conventions of notations, cardinality of sets $S, T, \ldots$ will be denoted by the corresponding lower case $s, t, \ldots$. We omit braces for sets, e.g., $\{k, m\}$, $N \setminus \{j\}$, will be written km, $N \setminus j$, etc. We also introduce for any $\emptyset \neq S \subseteq N$ the set

$$I_S := \{i \in I \mid \forall k, j \in S \; [i_k = i_j]\}. \tag{14}$$

We denote by i_S the value i_k for some $k \in S$, $i \in I_S$. The following properties are immediate.

(i) $I_k = I = 2^N$ for all $k \in N$.
(ii) Letting $I_\emptyset := 2^N$, $I : 2^N \to 2^{2^N}$ is a strictly antitone function.
(iii) For any $S, T \neq \emptyset$: $I_{S \cap T} \supseteq I_S \cup I_T \supseteq I_S \cap I_T \supseteq I_{S \cup T}$ and

$$I_{S \cap T} = I_S \cup I_T \quad iff \quad (\min_{K \in \{S,T\}} |K| = 1 \; or \; S \subseteq T \; or \; T \subseteq S) \tag{15}$$

$$I_S \cup I_T = I_S \cap I_T \quad iff \quad (\max_{K \in \{S,T\}} |K| = 1 \; or \; S = T) \tag{16}$$

$$I_S \cap I_T = I_{S \cup T} \quad iff \quad S \cap T \neq \emptyset. \tag{17}$$

Proof: (iii) If $|S| = 1$ and $S \cap T = \emptyset$, then $I_{S \cap T} = I_\emptyset = I = I_S = I_S \cup I_T$.
If $|S| = 1$ and $S \cap T \neq \emptyset$, then $S \cap T = S$ and $I_{S \cap T} = I_S = I = I_S \cup I_T$.
If $S \subseteq T$, then $I_{S \cap T} = I_S$, $I_T \subseteq I_S$ and $I_S \cup I_T = I_S$.
Suppose now that $\min_{K \in \{S,T\}} |K| > 1$, $S \not\subseteq T$ and $T \not\subseteq S$. Hence, if $S \cap T = \emptyset$, then $I_{S \cap T} = I_\emptyset = I \supset I_S \cup I_T$. On the other hand, if $S \cap T \neq \emptyset$, then also $I_{S \cap T} \supset I_S \cup I_T$.

If $|S| = |T| = 1$ or $S = T$, then $I_S = I_T$, and hence $I_S \cup I_T = I_S \cap I_T$.
If $\max_{K \in \{S,T\}} |K| > 1$ and $S \neq T$, then $I_S \cup I_T \supset I_S \cap I_T$.

If $S \cap T \neq \emptyset$, then $I_S \cap I_T = I_{S \cup T}$. On the other hand, if $S \cap T = \emptyset$, then $I_S \cap I_T \supset I_{S \cup T}$.
■

Let for each $S \subseteq N$ and $j \in N$

$$I_{S \to j} := \{i \in I \mid i_j = -i_S\} \tag{18}$$

$$I^*_{S \to j}(B) := \{i \in I \mid (Bi)_j = i_S = -i_j\}. \tag{19}$$

Of course, $I_{S \to j} = I^*_{S \to j}(B) = \emptyset$ for each $j \in S$ and $B \in \mathcal{B}$, and

$$|I_{S \to j}| = 2^{n-s}.$$

Definition 4 *Given $B \in \mathcal{B}$, for each $S \subseteq N$ and $j \in N$, the possibility influence index of coalition S on player j is defined as*

$$\overline{d}(B, S \to j) := \frac{|I^*_{S\to j}(B)|}{|I_{S\to j}|} = \frac{|\{i \in I \mid (Bi)_j = i_S = -i_j\}|}{2^{n-s}}. \tag{20}$$

For each $S \subseteq N$, $j \in N$, $B \in \mathcal{B}$

$$\overline{d}(B, S \to j) \in [0,1], \quad and \quad \overline{d}(B, S \to j) = 0 \; for \; j \in S.$$

3.2 The certainty influence index

Switching to another extreme way of calculating influence degree gives us the definitions of the certainty influence indices.

Definition 5 *Given $B \in \mathcal{B}$, for each $k, j \in N$ the certainty influence index of player k on player j is defined as*

$$\underline{d}(B, k \to j) := \frac{|\{i \in I^*_{k\to j}(B) \mid \forall p \neq k \; [i_p = i_j]\}|}{2}. \tag{21}$$

$\underline{d}(B, k \to j) \in \{0, \frac{1}{2}, 1\}$ expresses certainty of influence, i.e., it measures a degree of a certain influence player k has on player j. By analogy, we introduce

Definition 6 *Given $B \in \mathcal{B}$, for each $S \subseteq N$ and $j \in N$, the certainty influence index of coalition S on player j is given by*

$$\underline{d}(B, S \to j) := \frac{|\{i \in I^*_{S\to j}(B) \mid \forall p \notin S \; [i_p = i_j]\}|}{2}. \tag{22}$$

Again, for each $B \in \mathcal{B}$, $S \subseteq N$, $j \in N$

$$\underline{d}(B, j \to j) = 0, \quad \underline{d}(B, S \to j) = 0 \; for \; j \in S.$$

3.3 The weighted influence index

Finally, we propose a more general definition of the influence index. Let for each $i \in I$ and $k \in N$

$$n^*(i, k) := |\{m \in N \mid i_m = i_k\}| \geq 1 \tag{23}$$

measures simply the number of players with the same inclination as player k under i.

For each $k \in N$, $j \in N \setminus k$, and $i \in I_{k\to j}$, we introduce a *weight* $\alpha_i^{k\to j} \in [0,1]$ *of influence of player k on j under the inclination vector i*. We impose a *symmetry assumption*, i.e., we assume that $\alpha_i^{k\to j}$ depends solely on $n^*(i,k)$. We also assume that for each $k \in N$, $j \in N \setminus k$, there exists $i \in I_{k\to j}$ such that $\alpha_i^{k\to j} > 0$.

Definition 7 *Given $B \in \mathcal{B}$, for each $k \in N$, $j \in N \setminus k$, the weighted influence index of player k on player j is defined as*

$$d_\alpha(B, k \to j) := \frac{\sum_{i \in I^*_{k\to j}(B)} \alpha_i^{k\to j}}{\sum_{i \in I_{k\to j}} \alpha_i^{k\to j}} \in [0,1]. \tag{24}$$

The possibility and certainty influence indices are recovered as follows. For each $k \in N$, $j \in N \setminus k$ and $B \in \mathcal{B}$

$$\overline{d}(B, k \to j) = d_{\overline{\alpha}}(B, k \to j),\ where\ \overline{\alpha}_i^{k\to j} = 1\ for\ each\ i \in I_{k\to j} \tag{25}$$

$$\underline{d}(B, k \to j) = d_{\underline{\alpha}}(B, k \to j),\ where\ for\ each\ i \in I_{k\to j}$$
$$\underline{\alpha}_i^{k\to j} = \begin{cases} 1 \text{ if } \forall p \neq k\ [i_p = i_j] \\ 0 \qquad otherwise \end{cases}. \tag{26}$$

From among the whole family of the influence indices, apart from the two mentioned above, we specify another one, which we denote by d^* and refer to as an *equidistributed influence index*. Given $B \in \mathcal{B}$, for each $k \in N$, $j \in N \setminus k$

$$d^*(B, k \to j) := d_{\alpha^*}(B, k \to j),\ where\ \alpha_i^{*k\to j} := \frac{1}{2^{n^*(i,k)} - 1}\ for\ each\ i \in I_{k\to j}. \tag{27}$$

For the equidistributed influence index, we consider all cases of potential influence, i.e., for each $i \in I_{k\to j}$ we count the number of all p-player coalitions of the set $\{m \in N \mid i_k = i_m\}$, where $p = 1, ..., n^*(i, k)$, which gives

$$\sum_{p=1}^{n^*(i,k)} \binom{n^*(i,k)}{p} = 2^{n^*(i,k)} - 1.$$

By analogy, we can define the weighted influence index of a coalition on a player. Let for each $S \subseteq N$ and $i \in I_{S\to j}$

$$n^*(i, S) := |\{m \in N \mid \forall k \in S\ [i_m = i_k]\}| \geq s \tag{28}$$

where $n^*(i, S)$ is the number of players with the same inclination as players of S under $i \in I_{S\to j}$ (including the players from S).

For each $S \subseteq N$, $j \in N \setminus S$ and $i \in I_{S\to j}$, we introduce a *weight* $\alpha_i^{S\to j} \in [0, 1]$ *of influence of coalition* S *on* $j \notin S$ *under the inclination vector* $i \in I_{S\to j}$. As before, we assume that for each $S \subseteq N$ and $j \in N \setminus S$ there exists $i \in I_{S\to j}$ such that $\alpha_i^{S\to j} > 0$. Moreover, we impose the *symmetry assumption* that $\alpha_i^{S\to j}$ depends solely on $n^*(i, S)$, i.e., for each $S, S' \subseteq N$, $j \notin S$, $j' \notin S'$, $i \in I_{S\to j}$, $i' \in I_{S'\to j'}$,

$$if\ n^*(i, S) = n^*(i', S'),\ then\ \alpha_i^{S\to j} = \alpha_{i'}^{S'\to j'} \tag{29}$$

In particular, for each $S \subset N$, $j \notin S$ and $i \in I_{S\to j}$,

$$\alpha_i^{S\to j} = \alpha_{-i}^{S\to j}. \tag{30}$$

Definition 8 *Given $B \in \mathcal{B}$, for each $S \subseteq N$, $j \in N \setminus S$, the weighted influence index of coalition S on player j is defined as*

$$d_\alpha(B, S \to j) := \frac{\sum_{i \in I^*_{S\to j}(B)} \alpha_i^{S\to j}}{\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}} \in [0, 1]. \tag{31}$$

The possibility and certainty influence indices are recovered as follows. For each $S \subseteq N$, $j \in N \setminus S$ and $B \in \mathcal{B}$

$$\overline{d}(B, S \to j) = d_{\overline{\alpha}}(B, S \to j),\ where\ \overline{\alpha}_i^{S\to j} = 1\ for\ each\ i \in I_{S\to j} \tag{32}$$

$$\underline{d}(B, S \to j) = d_{\underline{\alpha}}(B, S \to j),\ where\ for\ each\ i \in I_{S\to j}$$
$$\underline{\alpha}_i^{S\to j} = \begin{cases} 1 \text{ if } \forall p \notin S\ [i_p = i_j] \\ 0 \qquad otherwise \end{cases}. \tag{33}$$

As before, apart from the possibility and certainty influence indices, we specify the equidistributed influence index. Given $B \in \mathcal{B}$, for each $S \subseteq N$, $j \in N \setminus S$

$$d^*(B, S \to j) = d_{\alpha^*}(B, S \to j),\ where\ \alpha_i^{*S\to j} = \frac{1}{2^{n^*(i,S)} - 1}\ for\ each\ i \in I_{S\to j}. \tag{34}$$

Definition 9 *Consider a family of weighted influence indices*

$$\mathcal{D}(B, j) := \{d_\alpha(B, S \to j)\}_{S \subseteq N\setminus j, S\neq\emptyset} \tag{35}$$

on j, with weights $\alpha_i^{S\to j}$, $i \in I$, $S \subseteq N \setminus j, S \neq \emptyset$. The family $\mathcal{D}(B, j)$ (or equivalently the family of their associate weights) is said to be situation invariant if for all $S \subseteq N \setminus j$, for all $i \in I^*_{S\to j}(B)$, $\alpha_i^{S\to j} = \alpha_i^{k\to j}$, for all $k \in S$.

Let us remark that:

(i) the possibility influence index is situation invariant, since all weights are equal to 1.
(ii) the certainty influence index is not situation invariant. Indeed, taking $S = \{k, m\}$ and $i \in I^*_{km\to j}(B)$, we have necessarily $i_k = i_m = -i_j$. Then $\underline{\alpha}_i^{k\to j} = \underline{\alpha}_i^{m\to j} = 0$, although $\underline{\alpha}_i^{km\to j} = 1$ for one such i.
(iii) the equidistributed influence index is situation invariant, since for any $i \in I^*_{S\to j}(B)$, $n^*(i, S) = n^*(i, k)$, for all $k \in S$.

Proposition 1 *Let $\mathcal{D}(B, j)$ be a situation invariant family of weighted influence indices on j. Then for any $S \subseteq N \setminus j$,*

$$\sum_{k\in S} d_\alpha(B, k \to j) \geq \frac{|S| \sum_{i\in I_{S\to j}} \alpha_i^{S\to j}}{\sum_{i\in I_{k\to j}} \alpha_i^{k\to j}} d_\alpha(B, S \to j). \tag{36}$$

Proof: For any $k \in S$, we have:

$$\begin{aligned}
d_\alpha(B, k \to j) &= \frac{\sum_{i\in I^*_{k\to j}(B)} \alpha_i^{k\to j}}{\sum_{i\in I_{k\to j}} \alpha_i^{k\to j}} \\
&= \frac{1}{\sum_{i\in I_{k\to j}} \alpha_i^{k\to j}} \Big[\sum_{i\in I^*_{S\to j}(B)} \alpha_i^{k\to j} + \sum_{K\subseteq S\setminus k, K\neq\emptyset} \sum_{\substack{\{i | i_m = i_j, m\in K \text{ and} \\ i_m = -i_j = (Bi)_j, m\in S\setminus K\}}} \alpha_i^{k\to j} \Big] \\
&= \frac{1}{\sum_{i\in I_{k\to j}} \alpha_i^{k\to j}} \Big[\sum_{i\in I^*_{S\to j}(B)} \alpha_i^{S\to j} + \epsilon \Big]
\end{aligned}$$

with $\epsilon \geq 0$. This being true for any $k \in S$, we get:

$$\sum_{k\in S} d_\alpha(B, k \to j) \geq \frac{|S|}{\sum_{i\in I_{k\to j}} \alpha_i^{k\to j}} \sum_{i\in I^*_{S\to j}(B)} \alpha_i^{S\to j}$$
$$= \frac{|S| \sum_{i\in I_{S\to j}} \alpha_i^{S\to j}}{\sum_{i\in I_{k\to j}} \alpha_i^{k\to j}} d_\alpha(B, S \to j).$$

■

As a particular case, we obtain for the possibility influence index:

$$\sum_{k\in S} \overline{d}(B, k \to j) \geq \frac{s}{2^{s-1}} \overline{d}(B, S \to j). \tag{37}$$

Also, for $S = \{k, m\}$, this reduces to the simple subadditive relation:

$$\overline{d}(B, k \to j) + \overline{d}(B, m \to j) \geq \overline{d}(B, km \to j). \tag{38}$$

Example 1 Let us add a few lines more about a relation between the influence index of a coalition and the sum of the influence indices of the members of that coalition. We notice that there exists a coalition S with $s \geq 3$ such that

$$\sum_{k\in S} \overline{d}(B, k \to j) < \overline{d}(B, S \to j), \tag{39}$$

and there exists a coalition S with $s \geq 3$ such that

$$\sum_{k\in S} \overline{d}(B, k \to j) > \overline{d}(B, S \to j). \tag{40}$$

In order to show this, we consider a four-player network in which player 4 is influenced by players 1, 2 and 3. Suppose player 4 will follow the others only if all players 1, 2 and 3 have the same inclination; otherwise he will decide according to his own inclination. In other words, function B is defined as follows:

$$\forall k \in \{1, 2, 3\}\ \forall i \in I\ [(Bi)_k = i_k]$$
$$(Bi)_4 = \begin{cases} i_1\ if\ i_1 = i_2 = i_3 \\ i_4 \quad otherwise \end{cases}.$$

Hence, we get

$$\forall k \in \{1, 2, 3\}\ [\ \overline{d}(B, k \to 4) = \frac{1}{4}\ ]$$
$$\overline{d}(B, 123 \to 4) = 1 > \sum_{k=1}^{3} \overline{d}(B, k \to 4).$$

We get a similar result for an n-player network with $n > 4$, where player n is influenced by all players $1, 2, ..., n-1$, and he will follow the others only if they all have the same inclination. If we define the influence function differently, for instance as B' such that player 4 will always follow the majority of the influencing players, then we get

$$\overline{d}(B', 123 \to 4) = 1 < \frac{9}{4} = \sum_{k=1}^{3} \overline{d}(B', k \to 4).$$

3.4 Influential null player

We introduce the following definition:

Definition 10 *Player $k \in N$ is said to be an influential null player under $B \in \mathcal{B}$ if*

$$\forall j \in N \setminus k \ \forall i \in I_{k \to j} \ [(Bi)_j = (Bi^{-k})_j], \tag{41}$$

where i^{-k} is given by

$$i_m^{-k} = \begin{cases} i_m & if\ m \neq k \\ -i_m & if\ m = k \end{cases} \quad for\ each\ m \in N. \tag{42}$$

Fact 1 *If (B-1), (B-2) are satisfied and $k \in N$ is an influential null player under $B \in \mathcal{B}$, then*

$$\forall j \in N \ [\underline{d}(B, k \to j) = 0]. \tag{43}$$

Proof: Let $k \in N$ be an influential null player under $B \in \mathcal{B}$ and let $j \in N \setminus k$. Let $i \in I_{k \to j}$ be such that $+1 = i_k = -i_j = -i_p$ for each $p \neq k$, and let $i' \in I_{k \to j}$ be such that $-1 = i'_k = -i'_j = -i'_p$ for each $p \neq k$. Hence, by virtue of axioms (B-1), (B-2), and (41),

$$(Bi)_j = (Bi^{-k})_j = (B(-1_N))_j = -1 = -i_k$$

$$(Bi')_j = (Bi'^{-k})_j = (B1_N)_j = +1 = -i'_k$$

Hence, $\underline{d}(B, k \to j) = 0$. ■

Let us consider the following monotonicity condition:

$$\forall i, i' \in I \ [i \leq i' \ \Rightarrow \ Bi \leq Bi'] \tag{44}$$

where

$$Bi \leq Bi' \iff \{k \in N \mid (Bi)_k = +1\} \subseteq \{k \in N \mid (Bi')_k = +1\}. \tag{45}$$

This condition will be violated, for instance, if there is a kind of 'opposite influence' ('My vote is (always) different from your inclination'). Nevertheless, one may suppose that in many situations the condition (44) holds.

Fact 2 *If $\widetilde{k} \in N$ is an influential null player under $B \in \mathcal{B}$, and condition (44) is satisfied, then for each $j \in N$*

$$\widetilde{k} = \arg \min_{k \in N \setminus j} d_\alpha(B, k \to j) \tag{46}$$

Proof: Let $\widetilde{k} \in N$ be an influential null player under $B \in \mathcal{B}$, and $j \in N \setminus \widetilde{k}$. Let us consider an arbitrary $k \in N \setminus \{\widetilde{k}, j\}$. Note that to each $i \in I_{\widetilde{k} \to j}$ corresponds a unique $i' \in I_{k \to j}$ such that

$$i'_k = i_{\widetilde{k}}, \quad i'_{\widetilde{k}} = i_k, \quad i'_m = i_m \ for\ each\ m \in N \setminus \{\widetilde{k}, k\}, \tag{47}$$

and reciprocally. Hence, by virtue of (29), for these $i \in I_{\widetilde{k}\to j}$ and $i' \in I_{k\to j}$ satisfying (47),

$$\alpha_i^{\widetilde{k}\to j} = \alpha_{i'}^{k\to j},$$

and therefore,

$$\sum_{i\in I_{\widetilde{k}\to j}} \alpha_i^{\widetilde{k}\to j} = \sum_{i\in I_{k\to j}} \alpha_i^{k\to j}.$$

Suppose k is also an influential null player under $B \in \mathcal{B}$. For $i \in I_{\widetilde{k}\to j}$ and $i' \in I_{k\to j}$ satisfying (47), if $i \neq i'$ (that is, if $i_{\widetilde{k}} = -i_k$), then

$$(Bi)_j = (Bi^{-\widetilde{k}})_j = (Bi'^{-k})_j = (Bi')_j,$$

and therefore $(Bi)_j = (Bi')_j$, which leads us to the conclusion that
either ($i \in I^*_{\widetilde{k}\to j}(B)$ and $i' \in I^*_{k\to j}(B)$) or ($i \notin I^*_{\widetilde{k}\to j}(B)$ and $i' \notin I^*_{k\to j}(B)$).
Since additionally $\alpha_i^{\widetilde{k}\to j} = \alpha_{i'}^{k\to j}$, we get

$$d_\alpha(B, \widetilde{k}\to j) = \frac{\sum_{i\in I^*_{\widetilde{k}\to j}(B)} \alpha_i^{\widetilde{k}\to j}}{\sum_{i\in I_{\widetilde{k}\to j}} \alpha_i^{\widetilde{k}\to j}} = \frac{\sum_{i\in I^*_{k\to j}(B)} \alpha_i^{k\to j}}{\sum_{i\in I_{k\to j}} \alpha_i^{k\to j}} = d_\alpha(B, k\to j).$$

Suppose k is not an influential null player under $B \in \mathcal{B}$. For $i \in I_{\widetilde{k}\to j}$ and $i' \in I_{k\to j}$ satisfying (47), if $i \neq i'$ (that is, if $i_{\widetilde{k}} = -i_k$) and $(Bi')_j \neq (Bi'^{-k})_j$, then from (44) we have

$$-i_j = i_{\widetilde{k}} = i'_k = -i'_j = (Bi')_j = -(Bi'^{-k})_j = -(Bi)_j,$$

which means that ($i \notin I^*_{\widetilde{k}\to j}(B)$ and $i' \in I^*_{k\to j}(B)$). Hence, summarizing,

$$d_\alpha(B, \widetilde{k}\to j) \leq d_\alpha(B, k\to j).$$

■

4 The influence functions

Let us study some properties of the influence functions $B \in \mathcal{B}$. Before we focus of the influence functions, we remark properties of some related concepts.

Definition 11 *Let $\emptyset \neq S \subseteq N$ and $B \in \mathcal{B}$. The set of followers of S under B is defined as*

$$F_B(S) := \{j \in N \mid \forall i \in I_S\ [(Bi)_j = i_S]\}. \tag{48}$$

The set of anti-followers of S under B is defined as

$$\overline{F}_B(S) := \{j \in N \mid \forall i \in I_S\ [(Bi)_j = -i_S]\}. \tag{49}$$

Letting $F_B(\emptyset) := \emptyset$, F_B is a mapping from 2^N to 2^N.
In general $F_B(S) \not\supseteq S$, although this would be a desirable property in general. If in addition $F_B(F_B(S)) = F_B(S)$ holds (not true in general), then F_B is a closure operator.

Definition 12 *An influence function $B \in \mathcal{B}$ is said to be strong if $F_B(S) \supseteq S$ for each $S \subseteq N$.*

Proposition 2 *Let $B \in \mathcal{B}$. Then the following holds:*

(i) Whenever $S \cap T = \emptyset$, $F_B(S) \cap F_B(T) = \emptyset$.
(ii) F_B is an isotone function. Consequently, if $F_B(N) = \emptyset$, then $F_B \equiv \emptyset$.
(iii) $F_B(N) = N$ iff both axioms (B-1) and (B-2) are satisfied.
(iv) If B is strong, then both axioms (B-1) and (B-2) are satisfied.
(v) For each $\widetilde{j} \in F_B(S) \setminus S$, $d_\alpha(B, S \to \widetilde{j}) = 1$.

Proof: (i) Since $S \cap T = \emptyset$, $I_S \cap I_T$ strictly includes $I_{S \cup T}$. Then there exist $i \in I_S \cap I_T$ such that $i_S = -i_T$. Hence if $j \in F_B(S) \cap F_B(T)$ the equality $(Bi)_j = i_S = i_T$ cannot hold for this i.
(ii) Take $S \subseteq S'$ and $j \in F_B(S)$. $i \in I_{S'}$ implies $i \in I_S$ by antitonicity, hence $(Bi)_j = i_S$ by hypothesis, and since $i_S = i_{S'}$, $j \in F_B(S')$.
(iii) Note that $I_N = \{1_N, -1_N\}$. $F_B(N) = N$ iff for each $j \in N$ $(B1_N)_j = +1$ and $(B(-1_N))_j = -1$ iff the axioms (B-1) and (B-2) are satisfied.
(iv) If at least one of the axioms is violated, then, $F_B(N) \neq N$. Hence, $N \neq F_B(N) \subseteq N$, which means that $F_B(N) \subset N$, and therefore B is not strong.
(v) Let $B \in \mathcal{B}$, $S \subset N$, $F_B(S) \neq \emptyset$, and $\widetilde{j} \in F_B(S) \setminus S$. Hence, for each $i \in I_S$, $(Bi)_{\widetilde{j}} = i_S$, and therefore $I^*_{S \to \widetilde{j}}(B) = I_{S \to \widetilde{j}}$, which means that $d_\alpha(B, S \to \widetilde{j}) = 1$. ■

Assume F_B is not identically the empty set. Then the *kernel* of B is the following collection of sets:

$$\mathcal{K}(B) := \{S \in 2^N \mid F_B(S) \neq \emptyset, \text{ and } S' \subset S \Rightarrow F_B(S') = \emptyset\}.$$

The kernel is well defined due to isotonicity. It it the set of "true" influential coalitions.

Definition 13 *Let S, T be two disjoint non empty subsets of N. B is said to be a purely influential function of S upon T if it satisfies for all $i \in I_S$:*

$$(Bi)j = \begin{cases} i_S, & \text{if } j \in T \\ i_j, & \text{otherwise.} \end{cases} \tag{50}$$

The set of such functions is denoted $\mathcal{B}_{S \to T}$.

Note that these functions are arbitrary on $I \setminus I_S$. What is the cardinality of $\mathcal{B}_{S \to T}$?
We have

$$|\mathcal{B}_{S \to T}| = 2^{n(2^s - 2)2^{n-s}} \tag{51}$$

In each $\mathcal{B}_{S \to T}$, there are 3 particular members. The minimal one is such that $Bi = -1_N$ for all $i \in I \setminus I_S$, the maximal one is such that $Bi = 1_N$ for all $i \in I \setminus I_S$. More interesting is the one which is the identity function on $I \setminus I_S$. We call it the *canonical pure influential function of S upon T*, and we denote it by $B_{S \to T}$.

Proposition 3 *Let S,T be two disjoint non empty subsets of N. Then the following holds:*

(i) For all $B \in \mathcal{B}_{S\to T}$, $F_B(S) = S \cup T$.

(ii) For each $B \in \mathcal{B}_{S\to T}$, $j \in N \setminus S$

$$d_\alpha(B, S \to j) = \begin{cases} 1 & \text{if } j \in T \\ 0 & \text{if } j \in N \setminus (S \cup T) \end{cases} \tag{52}$$

(iii) If B is strong, then for any two disjoint $S,T \subset N$, $B \notin \mathcal{B}_{S\to T}$.

Proof: (i) Take $t \in S \cup T$. If $t \in T$, then for any $i \in I_S$, $(Bi)_t = i_S$. If $t \in S$, then for any $i \in I_S$, $(Bi)_t = i_t = i_S$. Hence $t \in F_B(S)$. On the other hand, take $t \in F_B(S)$. Then for any $i \in I_S$, $(Bi)_t = i_S$, and hence $t \in S \cup T$.

(ii) Let $B \in \mathcal{B}_{S\to T}$. Then for each $i \in I_S$, $(Bi)_j = i_S$ for $j \in T$, and $(Bi)_j = i_j$ for $j \notin T$. Since $I_{S\to j} \subset I_S$, we have for each $i \in I_{S\to j}$ $(Bi)_j = i_S$ for $j \in T$, and $(Bi)_j = i_j$ for $j \notin T$. Hence, $I^*_{S\to j}(B) = I_{S\to j}$ for $j \in T$, and $I^*_{S\to j}(B) = \emptyset$ for $j \in N \setminus (S \cup T)$, and therefore if $j \in T$, then $d_\alpha(B, S \to j) = 1$, and if $j \in N \setminus (S \cup T)$, then $d_\alpha(B, S \to j) = 0$.

(iii) Suppose B is strong and there exist two disjoint sets $S,T \subset N$ such that $B \in \mathcal{B}_{S\to T}$. Hence, in particular for each $i \in I_S$ and $t \in T$, $(Bi)_t = i_S$. Take a particular $i \in I_S \cap I_T$ such that $i_S = -i_T$. Hence, we have $(Bi)_t = i_S = -i_t$ for each $t \in T$. On the other hand, since B is strong, we have for this particular $T \subset N$, $T \subseteq F_B(T)$, and consequently, for each $t \in T$ and $i \in I_T$, $(Bi)_t = i_T = i_t$. Contradiction.

■

We turn to influence functions and study their structure. $\mathcal{B}$ is the set of mappings from the Boolean lattice $(2^N, \subseteq)$ to $(2^N, \subseteq)$, and so is itself a Boolean lattice. We denote by $\leq$ the order relation, defined as $B \leq B'$ iff $Bi \leq B'i$ for all $i \in I$ (or with the set notation $BS \subseteq B'S$ for all $S \in 2^N$). Hence $\mathcal{B}$ is atomic, and its atoms are influence functions of the form

$$Bi := \begin{cases} (-1_{N\setminus j}, 1_j) & \text{if } i = i_0 \\ -1_N & \text{if } i \neq i_0 \end{cases}$$

for some $i_0 \in I$ and $j \in N$ (in set notation, $BS = \emptyset$ except for $S = S_0$, and $BS_0 = j$). Thus, the number of atoms is $n2^n$. Supremum and infimum are defined by $B \vee B' = B \cup B'$ and $B \wedge B' = B \cap B'$.

Next, we define several influence functions $B \in \mathcal{B}$ and investigate their properties. In particular, for each influence function analyzed, we determine the set of followers and the values of the weighted influence functions.

Some simple examples of influence functions are:

(i) *Majority function* - Let $n \geq t \geq \lfloor \frac{n}{2} \rfloor + 1$, and introduce

$$i^+ := |\{k \in N \mid i_k = +1\}| \tag{53}$$

We define $B \in \mathcal{B}$ such that for each $i \in I$

$$Bi := \begin{cases} 1_N & \text{if } \ i^+ \geq t \\ -1_N & \text{if } \ i^+ < t \end{cases} \tag{54}$$

(ii) $\widetilde{k} \in N$ *is a guru* - We define $B \in \mathcal{B}$ such that for each $i \in I$ and $j \in N$

$$(Bi)_j = i_{\widetilde{k}} \tag{55}$$

(iii) *The identity function (no influence)*, i.e., for each $i \in I$

$$Bi = i \tag{56}$$

(iv) *The reversal function* (systematic reversal of inclination; no clear phenomenon of influence), i.e., for each $i \in I$

$$Bi = -i \tag{57}$$

(v) *Order preserving functions*, i.e., for each $i, i' \in I$

$$if \ i \leq i', \ then \ Bi \leq Bi', \tag{58}$$

where

$$i \leq i' \iff \{k \in N \mid i_k = +1\} \subseteq \{k \in N \mid i'_k = +1\} \tag{59}$$

(vi) *Order reversing functions*, i.e., for each $i, i' \in I$

$$if \ i \leq i', \ then \ Bi \geq Bi' \tag{60}$$

(vii) Let $t \in (0, n]$. Functions satisfying for each $i \in I$

$$if \ i^+ \geq t, \ then \ i \leq Bi \tag{61}$$

(effect of mass psychology when i has a sufficient number of $+1$)
REMARK: Defining this with equivalence: $i \leq Bi \ iff \ i^+ \geq t$ would give a problem for $t \in (0, n]$: we would have $i \not\leq Bi$ for $i^+ < t$, but for $i = -1_N$, we have $i \leq Bi$.

(viii) Let $t \in [0, n)$. Functions satisfying for each $i \in I$

$$if \ i^+ \leq t, \ then \ i \geq Bi \tag{62}$$

(effect of the empty restaurant when i has a low number of $+1$)
REMARK: Defining this with equivalence: $i \geq Bi \ iff \ i^+ \leq t$ would give a problem for $t \in [0, n)$: we would have $i \not\geq Bi$ for $i^+ > t$, but for $i = 1_N$, we have $i \geq Bi$.

(ix) Let $t \in (0, n]$. Functions mixing the last two cases, i.e., satisfying for each $i \in I$

$$i \leq Bi \ iff \ i^+ \geq t, \ and \ i \geq Bi \ iff \ i^+ < t \tag{63}$$

Let us introduce for each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$:

$$I^+_{S \to j} := \{i \in I_{S \to j} \mid i_S = +1\} \tag{64}$$

$$I^-_{S \to j} := \{i \in I_{S \to j} \mid i_S = -1\} \tag{65}$$

and additionally,

$$I^+_{S \to j, >t} := \{i \in I^+_{S \to j} \mid i^+ > t\} \quad for \ t < n - 1 \tag{66}$$

$$I^+_{S \to j, \geq t} := \{i \in I^+_{S \to j} \mid i^+ \geq t\} \quad for \ t < n \tag{67}$$

$$I^-_{S \to j, <t} := \{i \in I^-_{S \to j} \mid i^+ < t\} \quad for \ t > 1. \tag{68}$$

We list some basic properties of the influence functions mentioned.

Proposition 4 *Let $n \geq t \geq \lfloor \frac{n}{2} \rfloor + 1$ and the influence function B be defined by (54). Then the following holds:*

(i) For each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$

$$d_\alpha(B, S \to j) = \begin{cases} 1 & \text{if } s \geq t \\ \frac{1}{2} & \text{if } s < t = n \\ \frac{\sum_{i \in I^+_{S\to j, \geq t}} \alpha_i^{S\to j} + \sum_{i \in I^-_{S\to j, < t}} \alpha_i^{S\to j}}{\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}} & \text{if } s < t < n \end{cases} \tag{69}$$

(ii) For each $S \subseteq N$

$$F_B(S) = \begin{cases} N & \text{if } s \geq t \\ \emptyset & \text{if } s < t \end{cases} \tag{70}$$

(iii) The kernel is $\mathcal{K}(B) = \{S \subseteq N \mid |S| = t\}$.

Proof: (i) Let $\emptyset \neq S \subseteq N$ be such that $s \geq t$, and $j \in N \setminus S$. If $i_S = +1$, then $i^+ \geq t$ and hence in particular $(Bi)_j = +1 = i_S$. If $i_S = -1$, then $i^+ < t$ and hence in particular $(Bi)_j = -1 = i_S$. This means that $I^*_{S\to j}(B) = I_{S\to j}$, and therefore $d_\alpha(B, S \to j) = 1$.

Let $\emptyset \neq S \subseteq N$ be such that $s < t$, $t = n$, and $j \in N \setminus S$. Hence, for each $i \in I \setminus \{1_N\}$, $Bi = -1_N$, and therefore $I^*_{S\to j}(B) = I^-_{S\to j}$. By virtue of (29), we have

$$d_\alpha(B, S \to j) = \frac{\sum_{i \in I^-_{S\to j}} \alpha_i^{S\to j}}{\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}} = \frac{\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}}{2\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}} = \frac{1}{2}$$

Let $\emptyset \neq S \subseteq N$ be such that $s < t$, $t < n$, and $j \in N \setminus S$. Hence, we get

$$d_\alpha(B, S \to j) = \frac{\sum_{i \in I^*_{S\to j}(B)} \alpha_i^{S\to j}}{\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}} = \frac{\sum_{i \in I^+_{S\to j, \geq t}} \alpha_i^{S\to j} + \sum_{i \in I^-_{S\to j, < t}} \alpha_i^{S\to j}}{\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}}$$

(ii) Let $S \subseteq N$ be such that $s \geq t$. Suppose that $F_B(S) \neq N$. This means that there exists $j \in N$ such that $j \notin F_B(S)$. Hence, there is $\widetilde{i} \in I_S$ such that $(B\widetilde{i})_j = -\widetilde{i}_S$. Suppose $\widetilde{i}_S = +1$. Hence, $i^+ \geq t$, and therefore for each $k \in N$, $(B\widetilde{i})_k = +1 = \widetilde{i}_S$. Suppose than that $\widetilde{i}_S = -1$. Hence, $i^+ < t$, and therefore for each $k \in N$, $(B\widetilde{i})_k = -1 = \widetilde{i}_S$. Contradiction.

Let $S \subseteq N$ be such that $s < t$. Suppose that $F_B(S) \neq \emptyset$. Let $\widetilde{j} \in F_B(S)$. Hence, for each $i \in I_S$, $(Bi)_{\widetilde{j}} = i_S$. Take $\widetilde{i} \in I_S$ such that $\widetilde{i}_S = -\widetilde{i}_k$ for each $k \notin S$. If $\widetilde{i}_S = +1$, then $\widetilde{i}^+ < t$, and hence for each $j \in N$, $(B\widetilde{i})_j = -1 = -\widetilde{i}_S$. If $\widetilde{i}_S = -1$, then $\widetilde{i}^+ \geq t$, and hence for each $j \in N$, $(B\widetilde{i})_j = +1 = -\widetilde{i}_S$. Contradiction.

(iii) By virtue of (70), we have the following. If $|S| < t$, then $F_B(S) = \emptyset$, and hence $S \notin \mathcal{K}(B)$. If $|S| = t$, then $F_B(S) = N$, but for each $S' \subset S$, $|S'| < t$, and therefore $F_B(S') = \emptyset$. Hence, $S \in \mathcal{K}(B)$. If $|S| > t$, then $F_B(S) = N$, and there exists $S' \subset S$ such that $|S'| \geq t$, which means that $F_B(S') = N$. Hence, $S \notin \mathcal{K}(B)$. ■

Proposition 5 *Let $\widetilde{k} \in N$ and the influence function B be defined by (55). Then the following holds:*

(i) For each $\emptyset \neq S \subseteq N$ and $j \in N \setminus (S \cup \{\widetilde{k}\})$

$$d_\alpha(B, S \to j) = \begin{cases} 1 & \text{if } \widetilde{k} \in S \\ \frac{1}{2} & \text{if } \widetilde{k} \notin S \end{cases} \tag{71}$$

(ii) For each $S \subseteq N$

$$F_B(S) = \begin{cases} N & \text{if } \widetilde{k} \in S \\ \emptyset & \text{if } \widetilde{k} \notin S \end{cases} \tag{72}$$

(iii) The kernel is $\mathcal{K}(B) = \{\widetilde{k}\}$.

(iv) B is the purely influential function of $\{\widetilde{k}\}$ upon $N \setminus \{\widetilde{k}\}$, i.e., $B \in \mathcal{B}_{\{\widetilde{k}\} \to N \setminus \{\widetilde{k}\}}$.

Proof: (i) Let $\emptyset \neq S \subseteq N$ be such that $\widetilde{k} \in S$, and $j \in N \setminus S$. Hence, for each $i \in I$, $(Bi)_j = i_{\widetilde{k}}$. This means in particular that $I^*_{S \to j}(B) = I_{S \to j}$, and hence $d_\alpha(B, S \to j) = 1$.

Let $\emptyset \neq S \subseteq N$ be such that $\widetilde{k} \notin S$, and $j \in N \setminus (S \cup \{\widetilde{k}\})$. Hence, if $i \in I_{S \to j}$ is such that $i_S = i_{\widetilde{k}}$, then $i \in I^*_{S \to j}(B)$, and if $i \in I_{S \to j}$ is such that $i_S = -i_{\widetilde{k}}$, then $i \notin I^*_{S \to j}(B)$. Hence, by virtue of (29), $d_\alpha(B, S \to j) = \frac{1}{2}$.

(ii) Let $S \subseteq N$ be such that $\widetilde{k} \in S$. Hence, for each $i \in I$, $i_S = i_{\widetilde{k}}$. Since $\widetilde{k}$ is a dictator, we have for each $j \in N$ and $i \in I$, $(Bi)_j = i_{\widetilde{k}}$. This means in particular that for each $j \in N$ and $i \in I_S$, $(Bi)_j = i_{\widetilde{k}} = i_S$, and therefore $F_B(S) = N$.

Let $S \subseteq N$ be such that $\widetilde{k} \notin S$. Suppose that $F_B(S) \neq \emptyset$. Let $\widetilde{j} \in F_B(S)$. Hence, for each $i \in I_S$, $(Bi)_{\widetilde{j}} = i_S$. Take $\widetilde{i} \in I_S$ such that $\widetilde{i}_S = -\widetilde{i}_k$. Hence, for each $j \in N$, $(B\widetilde{i})_j = \widetilde{i}_{\widetilde{k}} = -\widetilde{i}_S$, and in particular, $(B\widetilde{i})_{\widetilde{j}} = \widetilde{i}_{\widetilde{k}} = -\widetilde{i}_S$. Contradiction.

(iii) By virtue of (72), we have the following. If $\widetilde{k} \notin S$, then $F_B(S) = \emptyset$, and hence $S \notin \mathcal{K}(B)$. If $\widetilde{k} \in S$ and $|S| > 1$, then $F_B(S) = N = F_B(\widetilde{k})$, which means that $S \notin \mathcal{K}(B)$. Finally, since $F_B(\widetilde{k}) = N$ and $F_B(\emptyset) = \emptyset$, we have $\mathcal{K}(B) = \{\widetilde{k}\}$.

(iv) It comes immediately from (50) and (55). ■

Proposition 6 *Let the influence function B be defined by (56). Then the following holds:*

(i) B is the only influence function which is strong.

(ii) For each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$, $d_\alpha(B, S \to j) = 0$.

(iii) For each $S \subseteq N$, $F_B(S) = S$.

(iv) The kernel is $\mathcal{K}(B) = \{\{k\}, k \in N\}$.

Proof: (i) First of all, note that $\emptyset \subseteq F_B(\emptyset)$. Take an arbitrary $\emptyset \neq S \subseteq N$ and $j \in S$. By virtue of (56), for each $i \in I_S$, $(Bi)_j = i_j = i_S$, and therefore $j \in F_B(S)$ for each $j \in S$. Hence, $S \subseteq F_B(S)$ for each $S \subseteq N$.

Suppose now there is $\widetilde{B} \neq B$ which is strong. Hence, for each $S \subseteq N$, $S \subseteq F_{\widetilde{B}}(S)$.

Moreover, there is $\widetilde{i} \in I$ and $k \in N$ such that $(\widetilde{B}\widetilde{i})_k = -\widetilde{i}_k$. Take $S = \{j \in N \mid \widetilde{i}_j = \widetilde{i}_k\}$. Then, $k \in S$, but $k \notin F_{\widetilde{B}}(S)$, and therefore $S \not\subseteq F_{\widetilde{B}}(S)$. Hence, $\widetilde{B}$ is not strong.

(ii) If B satisfies (56), then for each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$, $I^*_{S\to j}(B) = \emptyset$. Hence, by virtue of (31), $d_\alpha(B, S \to j) = 0$.

(iii) Note that $S \subseteq F_B(S)$, because if $j \in S$, then in particular for each $i \in I_S$, $(Bi)_j = i_j = i_S$. Suppose $F_B(S) \not\subseteq S$. Hence, there is $k \notin S$ such that $k \in F_B(S)$, and therefore for each $i \in I_S$, $(Bi)_k = i_S$. Take $\widetilde{i} \in I_S$ such that $\widetilde{i}_S = -\widetilde{i}_k$. Then we have $(B\widetilde{i})_k = \widetilde{i}_S = -\widetilde{i}_k$, but $B\widetilde{i} = \widetilde{i}$. Contradiction.

(iv) By virtue of the point (iii), for each $S \subseteq N$, $F_B(S) = S$. Hence, for each $k \in N$, $\{k\} \in \mathcal{K}(B)$. For each $S \subseteq N$ such that $|S| > 1$, $F_B(S) = S$, but also for each S' such that $\emptyset \neq S' \subset S$, $F_B(S') = S' \neq \emptyset$. Hence $\mathcal{K}(B) = \{\{k\}, k \in N\}$. ■

Proposition 7 *Let the influence function B be defined by (57). Then the following holds:*

(i) None of the axioms (B-1), (B-2) is satisfied.
(ii) For each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$, $d_\alpha(B, S \to j) = 1$.
(iii) For each $S \subseteq N$, $F_B(S) = \emptyset$.
(iv) For any two disjoint $S, T \subseteq N$, $B \notin \mathcal{B}_{S\to T}$.

Proof: (i) From (57), $B1_N = -1_N$ and $B(-1_N) = 1_N$.

(ii) Take arbitrary $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$. If B is defined by (57), then $I^*_{S\to j}(B) = I_{S\to j}$. Hence, $d_\alpha(B, S \to j) = 1$.

(iii) Suppose there is $S \subseteq N$ such that $F_B(S) \neq \emptyset$, that is, there is $j \in F_B(S)$. Then, for each $i \in I_S$, $(Bi)_j = i_S$. Take $\widetilde{i} \in I_S$ such that $\widetilde{i}_S = \widetilde{i}_j$. Hence, $(B\widetilde{i})_j = \widetilde{i}_S = \widetilde{i}_j$, but $B\widetilde{i} = -\widetilde{i}$. Contradiction.

(iv) Take two arbitrary $S, T \subseteq N$ such that $S \cap T = \emptyset$. Note that $1_N \in I_S$ and $-1_N \in I_S$. By virtue of (57), $(Bi)_j = -i_j$ for each $j \in N$. Hence, $(B1_N)_j = -1$ and $(B(-1_N))_j = +1$ for each $j \in T$. ■

Proposition 8 *Let the influence function B be defined by (58). Then the following holds:*

(i) At least one of the axioms (B-1), (B-2) may be violated.
(ii) There is $B \in \mathcal{B}$ such that for some $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$, $d_\alpha(B, S \to j) = 1$.
(iii) There is $B \in \mathcal{B}$ such that for each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$, $d_\alpha(B, S \to j) = 0$.
(iv) There is $B \in \mathcal{B}$ such that $F_B(S) = S$ for each $S \subseteq N$.
There is $B \in \mathcal{B}$ such that $F_B(S) = N$ for some $S \subseteq N$.
There is $B \in \mathcal{B}$ such that $F_B(S) = \emptyset$ for some $S \subseteq N$.
(v) There exists $B \in \mathcal{B}$ such that for any two disjoint $S, T \subset N$, $B \notin \mathcal{B}_{S\to T}$.
(vi) There exists $B \in \mathcal{B}$ such that $B \in \mathcal{B}_{S\to T}$ for some $S, T \subset N$, $S \cap T = \emptyset$.

Proof: (i) Assume, for instance that $Bi = Bi'$ for each $i, i' \in I$. For the violation of only one axiom, (B-1) (respectively (B-2)), define for example $Bi = -1_N$ (respectively

$Bi = 1_N$). For the violation of both axioms, define for instance $(Bi)_j = +1$ if $j \leq \lfloor \frac{N}{2} \rfloor$, and $(Bi)_j = -1$ if $j > \lfloor \frac{N}{2} \rfloor$, for $n \geq 2$.

(ii) Define B as follows. There is $k \in N$ who is a guru, i.e., $(Bi)_j = i_k$ for each $i \in I$ and $j \in N$. Then, for each $S \subseteq N$ with $k \in S$ we have $I^*_{S \to j}(B) = I_{S \to j}$ for each $j \in N \setminus S$, and consequently, $d_\alpha(B, S \to j) = 1$.

(iii) Define B such that $Bi = i$ for each $i \in I$. By virtue of Proposition 6, (ii), we get $d_\alpha(B, S \to j) = 0$.

(iv) Take $Bi = i$ for each $i \in I$. Then $F_B(S) = S$ for each $S \subseteq N$.
Take now an arbitrary $k \in N$, and define B such that k is a guru. Then

$$F_B(S) = \begin{cases} N & \text{if } k \in S \\ \emptyset & \text{if } k \notin S. \end{cases}$$

(v) It results immediately from Proposition 3, (iv), since the function satisfying (56) is strong and also satisfies (58).

(vi) Take B from the proof of (ii), $S = \{k\}$ and $T = N \setminus k$. Hence, $B \in \mathcal{B}_{S \to T}$. ∎

Proposition 9 *Let the influence function B be defined by (60). Then the following holds:*

(i) At least one of the axioms (B-1), (B-2) is violated.
(ii) There is $B \in \mathcal{B}$ such that for each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$, $d_\alpha(B, S \to j) = 1$.
(iii) There is $B \in \mathcal{B}$ such that for some $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$, $d_\alpha(B, S \to j) = 0$.
(iv) For each $S \subseteq N$, $F_B(S) = \emptyset$.
(v) There exists $B \in \mathcal{B}$ such that for any two disjoint $S, T \subseteq N$, $B \notin \mathcal{B}_{S \to T}$.

Proof: (i) If both axioms are satisfied, then $B1_N = 1_N$ and $B(-1_N) = -1_N$, but then (60) is violated.

(ii) Define B such that $Bi = -i$ for each $i \in I$. By virtue of Proposition 7, (ii), we get $d_\alpha(B, S \to j) = 1$ for each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$.

(iii) Define B as follows. There is $k \in N$ such that for each $j \in N$ $(Bi)_j = -i_k$. Then, for each $S \subseteq N$ such that $k \in S$, $I^*_{S \to j}(B) = \emptyset$ for each $j \in N \setminus S$.

(iv) Suppose there is B satisfying (60) such that for a certain $S \subseteq N$, $F_B(S) \neq \emptyset$. Let $k \in F_B(S)$. Hence, for each $i \in I_S$, $(Bi)_k = i_S$. Take $i, i' \in I_S$ such that $i_S = i_k = -1$ and $i'_S = i'_k = +1$, and $i_j = i'_j$ for each each $j \in N \setminus (S \cup \{k\})$. Hence, $i \leq i'$, $(Bi)_k = -1$, $(Bi')_k = +1$, but this means that $Bi \not\geq Bi'$. Contradiction.

(v) It results immediately from Proposition 7, (iv), since (57) also satisfies (60). ∎

Proposition 10 *Let the influence function B be defined by (61). Then the following holds:*

(i) For each $t \in [0, n]$ there is $B \in \mathcal{B}$ such that for each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$,

$$d_\alpha(B, S \to j) = 0. \tag{73}$$

(ii) If $t \in \{0,1\}$, then for each $\emptyset \neq S \subseteq N$, $j \in N \setminus S$, and $B \in \mathcal{B}$,

$$d_\alpha(B, S \to j) \leq \frac{1}{2}. \tag{74}$$

Moreover, there is $B \in \mathcal{B}$ such that for each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$,

$$d_\alpha(B, S \to j) = \frac{1}{2}. \tag{75}$$

(iii) If $t > 1$, then for each $\emptyset \neq S \subseteq N$ such that $s > n - t$, and $j \in N \setminus S$, there exists $B \in \mathcal{B}$ such that

$$d_\alpha(B, S \to j) = 1.$$

(iv) If $t > 1$, then for each $\emptyset \neq S \subseteq N$ such that $s \leq n - t$, $j \in N \setminus S$, and $B \in \mathcal{B}$,

$$d_\alpha(B, S \to j) \leq \frac{\sum_{i \in I^+_{S \to j}} \alpha_i^{S \to j} + \sum_{i \in I^-_{S \to j, <t}} \alpha_i^{S \to j}}{\sum_{i \in I_{S \to j}} \alpha_i^{S \to j}}, \tag{76}$$

in particular,

$$\overline{d}(B, S \to j) \leq \frac{1}{2} + \frac{1}{2^{n-s}} \sum_{p=0}^{t-2} \binom{n-s-1}{p} \tag{77}$$

$$\underline{d}(B, S \to j) \in \{0, \frac{1}{2}\}.$$

Moreover, for each $\emptyset \neq S \subseteq N$ such that $s \leq n - t$ and $j \in N \setminus S$, there is $B \in \mathcal{B}$ such that

$$d_\alpha(B, S \to j) = \frac{\sum_{i \in I^+_{S \to j}} \alpha_i^{S \to j} + \sum_{i \in I^-_{S \to j, <t}} \alpha_i^{S \to j}}{\sum_{i \in I_{S \to j}} \alpha_i^{S \to j}}. \tag{78}$$

(v) For each $B \in \mathcal{B}$ and $S \subseteq N$,

$$F_B(S) \subseteq \begin{cases} N & \textit{if } s > n - t \textit{ and } t > 1 \\ S & \textit{if } s \leq n - t \end{cases} \tag{79}$$

Moreover, there is $B \in \mathcal{B}$ such that for each $S \subseteq N$, $F_B(S) = S$.

Proof: (i) Define B such that $Bi = i$ for each $i \in I$. By virtue of Proposition 6, (ii), we get $d_\alpha(B, S \to j) = 0$ for each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$.

(ii) Let B be defined by (61) and $t \in \{0,1\}$. By virtue of (30), for each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$

$$\sum_{i \in I_{S \to j}} \alpha_i^{S \to j} = 2 \sum_{i \in I^+_{S \to j}} \alpha_i^{S \to j}$$

Note that $I^*_{S \to j}(B) \cap I^-_{S \to j} = \emptyset$. Hence,

$$d_\alpha(B, S \to j) = \frac{\sum_{i \in I^*_{S \to j}(B)} \alpha_i^{S \to j}}{\sum_{i \in I_{S \to j}} \alpha_i^{S \to j}} \leq \frac{\sum_{i \in I^+_{S \to j}} \alpha_i^{S \to j}}{\sum_{i \in I_{S \to j}} \alpha_i^{S \to j}} = \frac{1}{2}.$$

Define B such that $Bi = 1_N$ for each $i \in I$. This B satisfies (61). Take arbitrary $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$. Note that $I^*_{S\to j}(B) = I^+_{S\to j}$. Hence, for this B

$$d_\alpha(B, S \to j) = \frac{1}{2}.$$

(iii) Let $t > 1$. Take arbitrary $\emptyset \neq S \subseteq N$ such that $s > n - t$, and $j \notin S$. Define B as follows:

$$(Bi)_k = \begin{cases} i_S & \text{if } i \in I_{S\to j} \text{ and } k = j \\ i_k & \text{otherwise} \end{cases}$$

Note that this B satisfies (61), because if $i \in I^-_{S\to j}$, then $i^+ < t$, and if $i \in I \setminus I^-_{S\to j}$, then (61) is satisfied. We have $d_\alpha(B, S \to j) = 1$.

(iv) Let B be defined by (61) and $t > 1$. Take arbitrary $\emptyset \neq S \subseteq N$ such that $s \leq n - t$, and $j \notin S$. Note that if $i \in I^-_{S\to j}$ is such that $i^+ \geq t$, then $i \notin I^*_{S\to j}(B)$. Hence,

$$d_\alpha(B, S \to j) \leq \frac{\sum_{i \in I^+_{S\to j}} \alpha_i^{S\to j} + \sum_{i \in I^-_{S\to j,<t}} \alpha_i^{S\to j}}{\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}}.$$

Moreover,

$$\overline{d}(B, S \to j) = \frac{|I^*_{S\to j}(B)|}{2^{n-s}} \leq \frac{|I^+_{S\to j}|}{2^{n-s}} + \frac{1}{2^{n-s}} \sum_{p=0}^{t-2} \binom{n-s-1}{p} = \frac{1}{2} + \frac{1}{2^{n-s}} \sum_{p=0}^{t-2} \binom{n-s-1}{p}$$

Moreover, $\underline{d}(B, S \to j) \neq 1$, because if $i \in I^-_{S\to j}$ is such that $i_k = +1$ for each $k \notin S$, then $i \notin I^*_{S\to j}(B)$. Hence, $\underline{d}(B, S \to j) \in \{0, \frac{1}{2}\}$.
Define B as follows. For each $k \neq j$, $(Bi)_k = i_k$ for $i \in I$, and

$$(Bi)_j = \begin{cases} i_S & \text{if } i \in I^+_{S\to j} \\ i_S & \text{if } i \in I^-_{S\to j} \text{ and } i^+ < t \\ i_j & \text{otherwise} \end{cases}$$

This B satisfies (61), and we get then

$$d_\alpha(B, S \to j) = \frac{\sum_{i \in I^+_{S\to j}} \alpha_i^{S\to j} + \sum_{i \in I^-_{S\to j,<t}} \alpha_i^{S\to j}}{\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}}.$$

(v) Let $t \in [0, n]$ and $s \leq n - t$. Suppose there is $B \in \mathcal{B}$ satisfying (61) such that $F_B(S) \not\subseteq S$ for a certain $S \subseteq N$. This means that $F_B(S) \neq \emptyset$, since $\emptyset \subseteq S$ for each S. Hence, there is $k \notin S$ such that $k \in F_B(S)$. This means that for each $i \in I_S$, $(Bi)_k = i_S$. Take $\widetilde{i} \in I_S$ such that $\widetilde{i}_S = -1$, $\widetilde{i}_k = +1$, and $\widetilde{i}^+ \geq t$. Such an $\widetilde{i}$ always exists, because $n - s \geq t$. We have $(B\widetilde{i})_k = \widetilde{i}_S = -1$, and therefore $\widetilde{i} \not\leq B\widetilde{i}$. But since $\widetilde{i}^+ \geq t$, we have $\widetilde{i} \leq B\widetilde{i}$. Contradiction.
If we take $Bi = i$ for each $i \in I$, then for each $S \subseteq N$, $F_B(S) = S$. ■

Proposition 11 *Let the influence function B be defined by (62). Then the following holds:*

(i) For each $t \in [0, n]$ there is $B \in \mathcal{B}$ such that for each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$

$$d_\alpha(B, S \to j) = 0.$$

(ii) If $t \in \{n-1, n\}$, then for each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$

$$d_\alpha(B, S \to j) \leq \frac{1}{2}.$$

Moreover, there is $B \in \mathcal{B}$ such that for each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$

$$d_\alpha(B, S \to j) = \frac{1}{2}.$$

(iii) If $t < n-1$, then for each $\emptyset \neq S \subseteq N$ such that $s > t$, and $j \in N \setminus S$, there exists $B \in \mathcal{B}$ such that

$$d_\alpha(B, S \to j) = 1.$$

(iv) If $t < n-1$, then for each $\emptyset \neq S \subseteq N$ such that $s \leq t$, and $j \in N \setminus S$

$$d_\alpha(B, S \to j) \leq \frac{\sum_{i \in I^-_{S \to j}} \alpha_i^{S \to j} + \sum_{i \in I^+_{S \to j, > t}} \alpha_i^{S \to j}}{\sum_{i \in I_{S \to j}} \alpha_i^{S \to j}} \tag{80}$$

in particular,

$$\overline{d}(B, S \to j) \leq \frac{1}{2} + \frac{1}{2^{n-s}} \sum_{p=t-s+1}^{n-s-1} \binom{n-s-1}{p} \tag{81}$$

$$\underline{d}(B, S \to j) \in \{0, \frac{1}{2}\}.$$

Moreover, for each $\emptyset \neq S \subseteq N$ such that $s \leq t$, and $j \in N \setminus S$, there exists $B \in \mathcal{B}$ such that

$$d_\alpha(B, S \to j) = \frac{\sum_{i \in I^-_{S \to j}} \alpha_i^{S \to j} + \sum_{i \in I^+_{S \to j, > t}} \alpha_i^{S \to j}}{\sum_{i \in I_{S \to j}} \alpha_i^{S \to j}}. \tag{82}$$

(v) For each $B \in \mathcal{B}$ and $S \subseteq N$,

$$F_B(S) \subseteq \begin{cases} N & \text{if } s > t \text{ and } t < n-1 \\ S & \text{if } s \leq t \end{cases} \tag{83}$$

Moreover, there is $B \in \mathcal{B}$ such that for each $S \subseteq N$, $F_B(S) = S$.

Proof: (i) Define B such that $Bi = i$ for each $i \in I$. By virtue of Proposition 6, (ii), we get $d_\alpha(B, S \to j) = 0$ for each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$.

(ii) Let B be defined by (62) and $t \in \{n-1, n\}$. By virtue of (30), for each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$

$$\sum_{i \in I_{S \to j}} \alpha_i^{S \to j} = 2 \sum_{i \in I^-_{S \to j}} \alpha_i^{S \to j}$$

Note that $I^*_{S\to j}(B) \cap I^+_{S\to j} = \emptyset$. Hence,

$$d_\alpha(B, S \to j) = \frac{\sum_{i\in I^*_{S\to j}(B)} \alpha_i^{S\to j}}{\sum_{i\in I_{S\to j}} \alpha_i^{S\to j}} \leq \frac{\sum_{i\in I^-_{S\to j}} \alpha_i^{S\to j}}{\sum_{i\in I_{S\to j}} \alpha_i^{S\to j}} = \frac{1}{2}.$$

Define B such that $Bi = -1_N$ for each $i \in I$. This B satisfies (62). Take arbitrary $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$. Note that $I^*_{S\to j}(B) = I^-_{S\to j}$. Hence, for this B

$$d_\alpha(B, S \to j) = \frac{1}{2}.$$

(iii) Let $t < n-1$. Take arbitrary $\emptyset \neq S \subseteq N$ such that $s > t$, and $j \notin S$. Define B as follows:

$$(Bi)_k = \begin{cases} i_S & \text{if } i \in I_{S\to j} \textit{ and } k = j \\ i_k & \text{otherwise} \end{cases}$$

Note that this B satisfies (62), because if $i \in I^+_{S\to j}$, then $i^+ > t$, and if $i \in I \setminus I^+_{S\to j}$, then (62) is satisfied. We have $d_\alpha(B, S \to j) = 1$.

(iv) Let B be defined by (62) and $t < n-1$. Take arbitrary $\emptyset \neq S \subseteq N$ such that $s \leq t$, and $j \notin S$. Note that if $i \in I^+_{S\to j}$ is such that $i^+ \leq t$, then $i \notin I^*_{S\to j}(B)$. Hence,

$$d_\alpha(B, S \to j) \leq \frac{\sum_{i\in I^-_{S\to j}} \alpha_i^{S\to j} + \sum_{i\in I^+_{S\to j,>t}} \alpha_i^{S\to j}}{\sum_{i\in I_{S\to j}} \alpha_i^{S\to j}}.$$

Moreover,

$$\begin{aligned} \overline{d}(B, S \to j) = \frac{|I^*_{S\to j}(B)|}{2^{n-s}} &\leq \frac{|I^-_{S\to j}(B)|}{2^{n-s}} + \frac{1}{2^{n-s}} \sum_{p=t-s+1}^{n-s-1} \binom{n-s-1}{p} \\ &= \frac{1}{2} + \frac{1}{2^{n-s}} \sum_{p=t-s+1}^{n-s-1} \binom{n-s-1}{p} \end{aligned}$$

$\underline{d}(B, S \to j) \neq 1$, because if $i \in I^+_{S\to j}$ is such that $i_k = -1$ for each $k \notin S$, then $i \notin I^*_{S\to j}(B)$. Hence, $\underline{d}(B, S \to j) \in \{0, \frac{1}{2}\}$.
Define B as follows. For each $k \neq j$, $(Bi)_k = i_k$ for $i \in I$, and

$$(Bi)_j = \begin{cases} i_S & \text{if } i \in I^-_{S\to j} \\ i_S & \text{if } i \in I^+_{S\to j} \textit{ and } i^+ > t \\ i_j & \text{otherwise} \end{cases}$$

This B satisfies (62), and we get then

$$d_\alpha(B, S \to j) = \frac{\sum_{i\in I^-_{S\to j}} \alpha_i^{S\to j} + \sum_{i\in I^+_{S\to j,>t}} \alpha_i^{S\to j}}{\sum_{i\in I_{S\to j}} \alpha_i^{S\to j}}.$$

(v) Let $\in [0,n]$ and $s \leq t$. Suppose there is $B \in \mathcal{B}$ satisfying (62) such that $F_B(S) \not\subseteq S$ for a certain $S \subseteq N$. This means that $F_B(S) \neq \emptyset$, since $\emptyset \subseteq S$ for each S. Hence, there is $k \notin S$ such that $k \in F_B(S)$. This means that for each $i \in I_S$, $(Bi)_k = i_S$. Take $\widetilde{i} \in I_S$ such that $\widetilde{i}_S = +1$, $\widetilde{i}_k = -1$, and $\widetilde{i}^+ \leq t$. Such an $\widetilde{i}$ always exists, because $s \leq t$. We have $(B\widetilde{i})_k = \widetilde{i}_S = +1$, and therefore $\widetilde{i} \not\geq B\widetilde{i}$. But since $\widetilde{i}^+ \leq t$, we have $\widetilde{i} \geq B\widetilde{i}$. Contradiction. If we take $Bi = i$ for each $i \in I$, then for each $S \subseteq N$, $F_B(S) = S$. ∎

Proposition 12 *Let the influence function B be defined by (63). Then the following holds:*

(i) For each $t \in (0,n]$ there is $B \in \mathcal{B}$ such that for each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$,

$$d_\alpha(B, S \to j) = 0. \tag{84}$$

(ii) If $t \in \{1,n\}$, then for each $\emptyset \neq S \subseteq N$, $j \in N \setminus S$, and $B \in \mathcal{B}$,

$$d_\alpha(B, S \to j) \leq \frac{1}{2}. \tag{85}$$

Moreover, there is $B \in \mathcal{B}$ such that for each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$,

$$d_\alpha(B, S \to j) = \frac{1}{2}. \tag{86}$$

(iii) If $1 < t < n$, then for each $\emptyset \neq S \subseteq N$ such that $s \geq t > n - s$, and $j \in N \setminus S$, there exists $B \in \mathcal{B}$ such that

$$d_\alpha(B, S \to j) = 1.$$

(iv) If $1 < t < n$, then for each $\emptyset \neq S \subseteq N$ such that $t \leq s \leq n-t$, $j \in N \setminus S$, and $B \in \mathcal{B}$,

$$d_\alpha(B, S \to j) \leq \frac{\sum_{i \in I^+_{S\to j}} \alpha_i^{S\to j} + \sum_{i \in I^-_{S\to j,<t}} \alpha_i^{S\to j}}{\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}} \tag{87}$$

in particular,

$$\overline{d}(B, S \to j) \leq \frac{1}{2} + \frac{1}{2^{n-s}} \sum_{p=0}^{t-2} \binom{n-s-1}{p} \tag{88}$$

$$\underline{d}(B, S \to j) \in \{0, \frac{1}{2}\}$$

Moreover, for each $\emptyset \neq S \subseteq N$ such that $t \leq s \leq n-t$ and $j \in N \setminus S$, there is $B \in \mathcal{B}$ such that

$$d_\alpha(B, S \to j) = \frac{\sum_{i \in I^+_{S\to j}} \alpha_i^{S\to j} + \sum_{i \in I^-_{S\to j,<t}} \alpha_i^{S\to j}}{\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}}. \tag{89}$$

(v) If $1 < t < n$, then for each $\emptyset \neq S \subseteq N$ such that $n - t < s < t$, and $j \in N \setminus S$

$$d_\alpha(B, S \to j) \leq \frac{\sum_{i \in I^-_{S\to j}} \alpha_i^{S\to j} + \sum_{i \in I^+_{S\to j,\geq t}} \alpha_i^{S\to j}}{\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}} \tag{90}$$

in particular,

$$\overline{d}(B, S \to j) \leq \frac{1}{2} + \frac{1}{2^{n-s}} \sum_{p=t-s}^{n-s-1} \binom{n-s-1}{p} \tag{91}$$

$$\underline{d}(B, S \to j) \in \{0, \frac{1}{2}\}$$

Moreover, for each $\emptyset \neq S \subseteq N$ *such that* $n - t < s < t$*, and* $j \in N \setminus S$*, there exists* $B \in \mathcal{B}$ *such that*

$$d_\alpha(B, S \to j) = \frac{\sum_{i \in I^-_{S\to j}} \alpha_i^{S\to j} + \sum_{i \in I^+_{S\to j, \geq t}} \alpha_i^{S\to j}}{\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}}. \tag{92}$$

(vi) If $1 < t < n$*, then for each* $\emptyset \neq S \subseteq N$ *such that* $s < t \leq n - s$*,* $j \in N \setminus S$*, and* $B \in \mathcal{B}$*,*

$$d_\alpha(B, S \to j) \leq \frac{\sum_{i \in I^+_{S\to j, \geq t}} \alpha_i^{S\to j} + \sum_{i \in I^-_{S\to j, < t}} \alpha_i^{S\to j}}{\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}} \tag{93}$$

in particular,

$$\overline{d}(B, S \to j) \leq \frac{1}{2^{n-s}} \sum_{p=t-s}^{n-s-1} \binom{n-s-1}{p} + \frac{1}{2^{n-s}} \sum_{p=0}^{t-2} \binom{n-s-1}{p} \tag{94}$$

$$\underline{d}(B, S \to j) = 0$$

Moreover, for each $\emptyset \neq S \subseteq N$ *such that* $s < t \leq n - s$ *and* $j \in N \setminus S$*, there is* $B \in \mathcal{B}$ *such that*

$$d_\alpha(B, S \to j) = \frac{\sum_{i \in I^+_{S\to j, \geq t}} \alpha_i^{S\to j} + \sum_{i \in I^-_{S\to j, < t}} \alpha_i^{S\to j}}{\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}}. \tag{95}$$

(vii) For each $B \in \mathcal{B}$ *and* $S \subseteq N$*,*

$$F_B(S) \subseteq \begin{cases} N & \text{if } s \geq t > n - s \\ S & \text{otherwise} \end{cases} \tag{96}$$

and additionally, if $s \geq t > n - s$*, then for each* $B \in \mathcal{B}$ *and* $S \subseteq N$*,* $S \subseteq F_B(S)$*. Moreover, there is* $B \in \mathcal{B}$ *such that for each* $S \subseteq N$*,* $F_B(S) = S$*.*

Proof: (i) Define B such that $Bi = i$ for each $i \in I$. By virtue of Proposition 6, (ii), we get $d_\alpha(B, S \to j) = 0$ for each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$.

(ii) Let B be defined by (63). By virtue of (30), for each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$

$$\sum_{i \in I_{S\to j}} \alpha_i^{S\to j} = 2 \sum_{i \in I^-_{S\to j}} \alpha_i^{S\to j} = 2 \sum_{i \in I^+_{S\to j}} \alpha_i^{S\to j}$$

Let $t = 1$. Note that $I^*_{S\to j}(B) \cap I^-_{S\to j} = \emptyset$. Hence,

$$d_\alpha(B, S \to j) = \frac{\sum_{i \in I^*_{S\to j}(B)} \alpha_i^{S\to j}}{\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}} \leq \frac{\sum_{i \in I^+_{S\to j}} \alpha_i^{S\to j}}{\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}} = \frac{1}{2}.$$

Define B such that $Bi = 1_N$ for each $i \in I$. This B satisfies (63). Take arbitrary $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$. Note that $I^*_{S\to j}(B) = I^+_{S\to j}$. Hence, for this B

$$d_\alpha(B, S \to j) = \frac{1}{2}.$$

Now, let $t = n$. Then $I^*_{S\to j}(B) \cap I^+_{S\to j} = \emptyset$. Hence,

$$d_\alpha(B, S \to j) = \frac{\sum_{i\in I^*_{S\to j}(B)} \alpha_i^{S\to j}}{\sum_{i\in I_{S\to j}} \alpha_i^{S\to j}} \leq \frac{\sum_{i\in I^-_{S\to j}} \alpha_i^{S\to j}}{\sum_{i\in I_{S\to j}} \alpha_i^{S\to j}} = \frac{1}{2}.$$

Define B such that $Bi = -1_N$ for each $i \in I$. This B satisfies (63). Take arbitrary $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$. Note that $I^*_{S\to j}(B) = I^-_{S\to j}$. Hence, for this B

$$d_\alpha(B, S \to j) = \frac{1}{2}.$$

(iii) Let $1 < t < n$. Take arbitrary $\emptyset \neq S \subseteq N$ such that $s \geq t > n - s$, and $j \notin S$. Define B such that for each $k \in N$

$$(Bi)_k = \begin{cases} i_S & \text{if } i \in I_{S\to j} \\ i_k & \text{otherwise} \end{cases}$$

Note that this B satisfies (63). We have $d_\alpha(B, S \to j) = 1$.

(iv) Let B be defined by (63) and $t > 1$. Take arbitrary $\emptyset \neq S \subseteq N$ such that $t \leq s \leq n-t$, and $j \notin S$. Note that if $i \in I^-_{S\to j}$ is such that $i^+ \geq t$, then $i \notin I^*_{S\to j}(B)$. Hence,

$$d_\alpha(B, S \to j) \leq \frac{\sum_{i\in I^+_{S\to j}} \alpha_i^{S\to j} + \sum_{i\in I^-_{S\to j,<t}} \alpha_i^{S\to j}}{\sum_{i\in I_{S\to j}} \alpha_i^{S\to j}}.$$

Moreover,

$$\overline{d}(B, S \to j) = \frac{|I^*_{S\to j}(B)|}{2^{n-s}} \leq \frac{|I^+_{S\to j}(B)|}{2^{n-s}} + \frac{1}{2^{n-s}} \sum_{p=0}^{t-2} \binom{n-s-1}{p} = \frac{1}{2} + \frac{1}{2^{n-s}} \sum_{p=0}^{t-2} \binom{n-s-1}{p}$$

$\underline{d}(B, S \to j) \neq 1$, because if $i \in I^-_{S\to j}$ is such that $i_k = +1$ for each $k \notin S$, then $i \notin I^*_{S\to j}(B)$. Hence, $\underline{d}(B, S \to j) \in \{0, \frac{1}{2}\}$.
Define B as follows. For each $k \neq j$, $(Bi)_k = i_k$ for $i \in I$, and

$$(Bi)_j = \begin{cases} i_S & \text{if } i \in I^+_{S\to j} \\ i_S & \text{if } i \in I^-_{S\to j} \textit{ and } i^+ < t \\ i_j & \text{otherwise} \end{cases}$$

This B satisfies (63), and we get then

$$d_\alpha(B, S \to j) = \frac{\sum_{i\in I^+_{S\to j}} \alpha_i^{S\to j} + \sum_{i\in I^-_{S\to j,<t}} \alpha_i^{S\to j}}{\sum_{i\in I_{S\to j}} \alpha_i^{S\to j}}.$$

(v) Let B be defined by (63) and $1 < t < n$. Take arbitrary $\emptyset \neq S \subseteq N$ such that $n - t < s < t$, and $j \notin S$. Note that if $i \in I^+_{S\to j}$ is such that $i^+ < t$, then $i \notin I^*_{S\to j}(B)$. Hence,

$$d_\alpha(B, S \to j) \leq \frac{\sum_{i \in I^-_{S\to j}} \alpha_i^{S\to j} + \sum_{i \in I^+_{S\to j, \geq t}} \alpha_i^{S\to j}}{\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}}.$$

Moreover,

$$\begin{aligned}\overline{d}(B, S \to j) = \frac{|I^*_{S\to j}(B)|}{2^{n-s}} &\leq \frac{|I^-_{S\to j}(B)|}{2^{n-s}} + \frac{1}{2^{n-s}} \sum_{p=t-s}^{n-s-1} \binom{n-s-1}{p} \\ &= \frac{1}{2} + \frac{1}{2^{n-s}} \sum_{p=t-s}^{n-s-1} \binom{n-s-1}{p}\end{aligned}$$

$\underline{d}(B, S \to j) \neq 1$, because if $i \in I^+_{S\to j}$ is such that $i_k = -1$ for each $k \notin S$, then $i \notin I^*_{S\to j}(B)$. Hence, $\underline{d}(B, S \to j) \in \{0, \frac{1}{2}\}$.
Define B as follows. For each $k \neq j$, $(Bi)_k = i_k$ for $i \in I$, and

$$(Bi)_j = \begin{cases} i_S & \text{if } i \in I^-_{S\to j} \\ i_S & \text{if } i \in I^+_{S\to j} \textit{ and } i^+ \geq t \\ i_j & \text{otherwise} \end{cases}$$

This B satisfies (63), and we get then

$$d_\alpha(B, S \to j) = \frac{\sum_{i \in I^-_{S\to j}} \alpha_i^{S\to j} + \sum_{i \in I^+_{S\to j, \geq t}} \alpha_i^{S\to j}}{\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}}.$$

(vi) Let B be defined by (63) and $1 < t < n$. Take arbitrary $\emptyset \neq S \subseteq N$ such that $s < t \leq n - s$, and $j \notin S$. Note that if $i \in I^+_{S\to j}$ is such that $i^+ < t$, then $i \notin I^*_{S\to j}(B)$, and if $i \in I^-_{S\to j}$ is such that $i^+ \geq t$, then $i \notin I^*_{S\to j}(B)$. Hence,

$$d_\alpha(B, S \to j) \leq \frac{\sum_{i \in I^+_{S\to j, \geq t}} \alpha_i^{S\to j} + \sum_{i \in I^-_{S\to j, < t}} \alpha_i^{S\to j}}{\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}}.$$

Moreover,

$$\overline{d}(B, S \to j) = \frac{|I^*_{S\to j}(B)|}{2^{n-s}} \leq \frac{1}{2^{n-s}} \sum_{p=t-s}^{n-s-1} \binom{n-s-1}{p} + \frac{1}{2^{n-s}} \sum_{p=0}^{t-2} \binom{n-s-1}{p}$$

$\underline{d}(B, S \to j) = 0$, because if $i \in I^+_{S\to j}$ is such that $i_k = -1$ for each $k \notin S$, then $i \notin I^*_{S\to j}(B)$, and if $i \in I^-_{S\to j}$ is such that $i_k = +1$ for each $k \notin S$, then $i \notin I^*_{S\to j}(B)$.
Define B as follows. For each $k \neq j$, $(Bi)_k = i_k$ for $i \in I$, and

$$(Bi)_j = \begin{cases} i_S & \text{if } i \in I^+_{S\to j} \textit{ and } i^+ \geq t \\ i_S & \text{if } i \in I^-_{S\to j} \textit{ and } i^+ < t \\ i_j & \text{otherwise} \end{cases}$$

This B satisfies (63), and we get then

$$d_\alpha(B, S \to j) = \frac{\sum_{i \in I^+_{S\to j,\geq t}} \alpha_i^{S\to j} + \sum_{i \in I^-_{S\to j,<t}} \alpha_i^{S\to j}}{\sum_{i \in I_{S\to j}} \alpha_i^{S\to j}}.$$

(vii) Let $s < t$ or $t \leq n - s$. Suppose there is $B \in \mathcal{B}$ satisfying (63) such that $F_B(S) \not\subseteq S$ for a certain $S \subseteq N$. This means that $F_B(S) \neq \emptyset$, since $\emptyset \subseteq S$ for each S. Hence, there is $k \notin S$ such that $k \in F_B(S)$. This means that for each $i \in I_S$, $(Bi)_k = i_S$.
Let $s < t$. Take $\widetilde{i} \in I_S$ such that $\widetilde{i}_S = +1$, $\widetilde{i}_k = -1$, and $\widetilde{i}^+ < t$. Such an $\widetilde{i}$ always exists, because $s < t$. We have $(B\widetilde{i})_k = \widetilde{i}_S = +1$, and therefore $\widetilde{i} \not\geq B\widetilde{i}$. But since $\widetilde{i}^+ < t$, we have $\widetilde{i} \geq B\widetilde{i}$. Contradiction.
Let $t \leq n - s$. Take $\widetilde{i} \in I_S$ such that $\widetilde{i}_S = -1$, $\widetilde{i}_k = +1$, and $\widetilde{i}^+ \geq t$. Again, such an $\widetilde{i}$ always exists, because $t \leq n - s$. We have $(B\widetilde{i})_k = \widetilde{i}_S = -1$, and therefore $\widetilde{i} \not\leq B\widetilde{i}$. But since $\widetilde{i}^+ \geq t$, we have $\widetilde{i} \leq B\widetilde{i}$. Contradiction.
Let $s \geq t > n - s$. Suppose there is $B \in \mathcal{B}$ such that $S \not\subseteq F_B(S)$ for a certain $S \subseteq N$. This means that there $k \in S$ such that $k \notin F_B(S)$, and therefore there is $\widetilde{i} \in I_S$ such that $(B\widetilde{i})_k = -\widetilde{i}_S$. If $\widetilde{i}_S = +1$, then $(B\widetilde{i})_k = -1$, and hence $\widetilde{i} \not\leq B\widetilde{i}$. But since $s \geq t$, we have $i^+ \geq t$, and therefore $\widetilde{i} \leq B\widetilde{i}$. Hence, $\widetilde{i}_S = -1$, and $(B\widetilde{i})_k = +1$. But this means that $\widetilde{i} \not\geq B\widetilde{i}$. Since $t > n - s$, we have $i^+ < t$, and therefore $\widetilde{i} \geq B\widetilde{i}$. Contradiction.

If we take $Bi = i$ for each $i \in I$, then for each $S \subseteq N$, $F_B(S) = S$.

■

5 Success, decisiveness and the revised Hoede-Bakker index

Let us consider the 'power part' of the global index. First of all, we adopt the following three axioms on $gd \in \mathcal{G}$:

AXIOM (G-1):

$$\forall i, i' \in I \; [Bi \leq Bi' \;\Rightarrow\; gd(Bi) \leq gd(Bi')], \tag{97}$$

where

$$Bi \leq Bi' \iff \{k \in N \mid (Bi)_k = +1\} \subseteq \{k \in N \mid (Bi')_k = +1\} \tag{98}$$

AXIOM (G-2):

$$gd(1_N) = +1 \tag{99}$$

AXIOM (G-3):

$$gd(-1_N) = -1. \tag{100}$$

Now, after separating the two functions B and gd, we define *Success*, *Failure*, *Luck* and *Decisiveness* of a player starting from the final decision of the player, not as before from the inclination. For instance, a player is said to be *successful* if his decision coincides with the group decision.

Given a probability distribution $p : I \to [0, 1]$ over all inclination vectors, and $B \in \mathcal{B}$, we define $p_B = p \circ B^{-1}$ on I (probability of the decision vectors).

Since some decision vectors may never appear after applying the influence function, we define now the group decision function $gd : I \rightarrow \{+1, -1\}$ on the set of all n-vectors, assigning (as before) the value $+1$ if the group decision is 'yes', and -1 if the group decision is 'no'. Moreover, for $b \in I$ and $k \in N$, we define $b^{-k} \in I$ by

$$b_j^{-k} = \begin{cases} b_j & if\ j \neq k \\ -b_j & if\ j = k \end{cases}. \tag{101}$$

Definition 14 *Given $gd \in \mathcal{G}$, $p_B : I \rightarrow [0,1]$, we define for each $k \in N$*

– *Player k's Success*

$$SUC_k(gd, p_B) := Prob(k\ is\ successful) = \sum_{\{b \in I | b_k = gd(b)\}} p_B(b) \tag{102}$$

– *Player k's Failure*

$$FAIL_k(gd, p_B) := Prob(k\ fails) = \sum_{\{b \in I | b_k = -gd(b)\}} p_B(b) \tag{103}$$

– *Player k's Decisiveness*

$$DEC_k(gd, p_B) := Prob(k\ is\ decisive) = \sum_{\{b \in I | b_k = gd(b) = -gd(b^{-k})\}} p_B(b) \tag{104}$$

– *Player k's Luck*

$$LUCK_k(gd, p_B) := Prob(k\ is\ lucky) = \sum_{\{b \in I | b_k = gd(b) = gd(b^{-k})\}} p_B(b) \tag{105}$$

According to Barry [2], the following relation between Success, Luck, and Decisiveness holds:

$$Success\ =\ Decisiveness\ +\ Luck,$$

and in our case, we have for each $k \in N$, p_B, and $gd \in \mathcal{G}$

$$SUC_k(gd, p_B) = DEC_k(gd, p_B) + LUCK_k(gd, p_B) \tag{106}$$

$$SUC_k(gd, p_B) = 1 - FAIL_k(gd, p_B). \tag{107}$$

The revised Hoede-Bakker index looks now as follows:

Definition 15 *Given $B \in \mathcal{B}$, $gd \in \mathcal{G}$, $p : I \rightarrow [0,1]$, we define for each $k \in N$*

$$\phi_k(B, gd, p) := \sum_{\{i | (Bi)_k = +1\}} p(i) gd(Bi) - \sum_{\{i | (Bi)_k = -1\}} p(i) gd(Bi). \tag{108}$$

Fact 3 *Given $B \in \mathcal{B}$, $gd \in \mathcal{G}$, and $p : I \rightarrow [0,1]$, for each $k \in N$*

$$\phi_k(B, gd, p) = SUC_k(gd, p_B) - FAIL_k(gd, p_B). \tag{109}$$

Proof: Let $B \in \mathcal{B}$, $gd \in \mathcal{G}$, $p : I \to [0,1]$, and $k \in N$. From (108), we have

$$\begin{aligned}\phi_k(B, gd, p) &= \sum_{\{i|(Bi)_k=+1\}} p(i)gd(Bi) - \sum_{\{i|(Bi)_k=-1\}} p(i)gd(Bi)\\ &= \sum_{\{i|(Bi)_k=gd(Bi)\}} p(i) - \sum_{\{i|(Bi)_k=-gd(Bi)\}} p(i)\end{aligned}$$

By virtue of (102) and (103),

$$\begin{aligned}SUC_k(gd, p_B) - FAIL_k(gd, p_B) &= \sum_{\{b\in I|b_k=gd(b)\}} p_B(b) - \sum_{\{b\in I|b_k=-gd(b)\}} p_B(b)\\ &= \sum_{\{b\in B(I)|b_k=gd(b)\}} \sum_{\{i|Bi=b\}} p(i) - \sum_{\{b\in B(I)|b_k=-gd(b)\}} \sum_{\{i|Bi=b\}} p(i)\\ &= \sum_{\{i|(Bi)_k=gd(Bi)\}} p(i) - \sum_{\{i|(Bi)_k=-gd(Bi)\}} p(i) = \psi_k(B, gd, p)\end{aligned}$$

■

Let us notice that while ϕ is equal to '*Success* − *Failure*' under the new definition of being successful (based on decisions, not as before on inclinations), in general it is not equal to '*Decisiveness*' anymore.

6 The global index

There is a trivial relation between the 'influence part' and the 'power part' of the global index if all inclination vectors are equally probable. Let $p^* : I \to [0,1]$ be the uniform distribution, i.e.,

$$\forall i \in I\ [p^*(i) = \frac{1}{2^n}]. \tag{110}$$

Assuming that axioms (B-1) and (B-2) are satisfied, the following fact holds:

Fact 4 *If $\overline{d}(B, k \to j) = 0$ for each $k \in N$, then $\phi_j(B, gd, p^*) = GHB_j(B, gd)$.*

Proof: Let $\overline{d}(B, k \to j) = 0$ for each $k \in N$. This means that for each $k \in N$ $|\{i \in I \mid (Bi)_j = i_k = -i_j\}| = 0$. Hence, for each $k \in N$ and $i \in I$, if $i_k = -i_j$, then $i_j = (Bi)_j$. Moreover, by virtue of (B-1) and (B-2), $B1_N = 1_N$ and $B(-1_N) = -1_N$. Hence

$$\begin{aligned}\phi_j(B, gd, p^*) &= \frac{1}{2^n}\left(\sum_{\{i|(Bi)_j=+1\}} gd(Bi) - \sum_{\{i|(Bi)_j=-1\}} gd(Bi)\right)\\ &= \frac{1}{2^n}\left(\sum_{\{i|i_j=+1\}} gd(Bi) - \sum_{\{i|i_j=-1\}} gd(Bi)\right) = GHB_j(B, gd).\end{aligned}$$

■

Example 2 In order to illustrate the notions introduced in the paper, let us consider the following example. We have a three-actor family network in which player 1 (child) is influenced by his mother and his father (players 2 and 3, respectively). If the parents have the same inclination, the child will follow them, but if their inclinations differ from each other, player 1 will decide according to his own inclination. The family has to decide for a long Sunday bicycle trip, but since the weather happens to be quite risky, the actors are not that enthusiastic to decide for the trip. Moreover, a new attractive computer game, a romance just bought and looking very interesting, and a telecast of an important football match are of importance when making the decision. The inclinations of the players to say 'yes' are independent of each other and their probabilities are equal to $\frac{1}{2}$, $\frac{1}{3}$, and 0, for the child, the mother and the father, respectively. The parents try not to discriminate their child in family decision-making, and it is agreed that the family decides for the trip if at least two family members say 'yes'. Table 1 presents the probability distribution over all inclination vectors, and the decision vectors, while Table 2 shows the probability distribution over all decision vectors, and the group decisions.

Table 1. The inclination and decision vectors

$i \in I$	$(1,1,1)$	$(1,1,-1)$	$(1,-1,1)$	$(-1,1,1)$	$(1,-1,-1)$	$(-1,1,-1)$	$(-1,-1,1)$	$(-1,-1,-1)$
$p(i)$	0	$\frac{1}{6}$	0	0	$\frac{1}{3}$	$\frac{1}{6}$	0	$\frac{1}{3}$
$B(i)$	$(1,1,1)$	$(1,1,-1)$	$(1,-1,1)$	$(1,1,1)$	$(-1,-1,-1)$	$(-1,1,-1)$	$(-1,-1,1)$	$(-1,-1,-1)$

Table 2. The group decision

$b \in B(I)$	$(1,1,1)$	$(1,1,-1)$	$(1,-1,1)$	$(-1,1,-1)$	$(-1,-1,1)$	$(-1,-1,-1)$
$p_B(b)$	0	$\frac{1}{6}$	0	$\frac{1}{6}$	0	$\frac{2}{3}$
$gd(b)$	$+1$	$+1$	$+1$	-1	-1	-1

Moreover, $gd(-1,1,1) = +1$, and $gd(1,-1,-1) = -1$.
Based on the given information, we get the following:

$$\forall k, j \in \{1,2,3\} \ [\underline{d}(B, k \to j) = 0]$$

$$\forall d \in \{\overline{d}, d^*\} \ [d(B, 1 \to 2) = d(B, 1 \to 3) = d(B, 2 \to 3) = d(B, 3 \to 2) = 0]$$

$$\overline{d}(B, 2 \to 1) = \overline{d}(B, 3 \to 1) = \frac{1}{2}, \quad d^*(B, 2 \to 1) = d^*(B, 3 \to 1) = \frac{1}{4}$$

$$\forall d \in \{\overline{d}, \underline{d}, d^*\} \ [d(B, 12 \to 3) = d(B, 13 \to 2) = 0]$$

$$\overline{d}(B, 23 \to 1) = \underline{d}(B, 23 \to 1) = d^*(B, 23 \to 1) = 1$$

$$F_B(1) = \emptyset, \quad F_B(2) = \{2\}, \quad F_B(3) = \{3\}$$

$$F_B(12) = \{1,2\}, \quad F_B(13) = \{1,3\}, \quad F_B(23) = \{1,2,3\}$$

$$\mathcal{K}(B) = \{\{2\},\{3\}\}, \quad B = B_{23 \to 1}$$

Note that our influence function happens to be the canonical pure influential function of the parents upon the child.

$$SUC_1(gd,p_B)=1,\quad SUC_2(gd,p_B)=SUC_3(gd,p_B)=\frac{5}{6}$$

$$FAIL_1(gd,p_B)=0,\quad FAIL_2(gd,p_B)=FAIL_3(gd,p_B)=\frac{1}{6}$$

$$DEC_1(gd,p_B)=\frac{1}{3},\quad DEC_2(gd,p_B)=DEC_3(gd,p_B)=\frac{1}{6}$$

$$LUCK_1(gd,p_B)=LUCK_2(gd,p_B)=LUCK_3(gd,p_B)=\frac{2}{3}$$

$$\phi_1(B,gd,p)=1,\quad \phi_2(B,gd,p)=\phi_3(B,gd,p)=\frac{2}{3}$$

7 Conclusions

The improvement brought in this paper emphasizes the role of the influence function in the Hoede-Bakker index. The global form of the index proposed here fully takes into account the mutual influence among players. In particular, we define the possibility influence index which takes into account any possibility of influence, and the certainty influence index which expresses certainty of influence. These indices are special cases of the weighted influence index. To the best of our knowledge, such influence indices have not been proposed before, and seem to be a very useful tool, in particular, in the theory of coalition and alliance formation, negotiations, and more generally multi-agent systems.

There are still several other improvements we would like to bring to this framework in the future research. One of the new research lines may be to introduce dynamic aspects. The framework analyzed here is, in fact, a decision process after a single step of mutual influence. In reality, the mutual influence does not stop necessarily after one step but may iterate. We propose to study the behavior of the series $Bi, B^2i, ..., B^ni$; to find convergence conditions, and to study the corresponding decisional power index.

Another natural improvement would be to enlarge the set of possible decisions. The original framework considers only a yes-no decision in a voting situation. One may enlarge this to a yes-no-abstention scheme (ternary voting games, [7]) or, if one escapes from voting situations, to multi-choice games [12], where each player has a totally ordered set of possible actions, and more generally to games on product lattices [9].

References

[1] J. Banzhaf. Weighted voting doesn't work: a mathematical analysis. *Rutgers Law Review* 19: 317-343, 1965.

[2] B. Barry. Is it better to be powerful or lucky?, Part I and Part II. *Political Studies* 28: 183-194, 338-352, 1980.

[3] J.S. Coleman. Control of collectivities and the power of a collectivity to act. In: Lieberman B. (ed.) *Social Choice*, New York: Gordon and Breach, 269-300, 1971.

[4] J.S. Coleman. *Individual Interests and Collective Action: Selected Essays*, Cambridge University Press, 1986.

[5] J. Deegan and E.W. Packel. A new index of power for simple n-person games. *International Journal of Game Theory* 7: 113-123, 1978.

[6] P. Dubey and L.S. Shapley. Mathematical properties of the Banzhaf power index. *Mathematics of Operations Research* 4: 99-131, 1979.

[7] D. Felsenthal and M. Machover. Ternary voting games. *International Journal of Game Theory* 26: 335-351, 1997.
[8] D. Felsenthal and M. Machover. The Measurement of Voting Power: Theory and Practice, Problems and Paradoxes, London: Edward Elgar Publishers, 1998.
[9] M. Grabisch. The Shapley value for games on lattices (l-fuzzy games). In *International Conference on Fuzzy Sets and Soft Computing in Economics and Finance*, St. Petersburg, Russia, June 2004.
[10] C. Hoede and R. Bakker. A theory of decisional power. *Journal of Mathematical Sociology* 8: 309-322, 1982.
[11] M.J. Holler and E.W. Packel. Power, luck and the right index. *Journal of Economics* 43: 21-29, 1983.
[12] C.R. Hsiao and T.E.S. Raghavan. Shapley value for multichoice cooperative games. *Games and Economic Behavior* 5: 240-256, 1993.
[13] R.J. Johnston. On the measurement of power: some reactions to Laver. *Environment and Planning A* 10: 907-914, 1978.
[14] T. König and Bräuninger. The inclusiveness of European decision rules. *Journal of Theoretical Politics* 10: 125-142, 1998.
[15] A. Laruelle and F. Valenciano. Assessing success and decisiveness in voting situations. *Social Choice and Welfare* 24: 171-197, 2005.
[16] L.S. Penrose. The elementary statistics of majority voting. *Journal of the Royal Statistical Society* 109: 53-57, 1946.
[17] D. Rae. Decision-rules and individual values in constitutional choice. *American Political Science Review* 63: 40-56, 1969.
[18] A. Rusinowska. The not-preference-based Hoede-Bakker index. GATE Working Paper 2007-4.
[19] A. Rusinowska and H. De Swart. Generalizing and modifying the Hoede-Bakker index. In: De Swart H. et al. (Eds.) *Theory and Applications of Relational Structures as Knowledge Instruments*, No 2. Springer's Lecture Notes in Artificial Intelligence LNAI 4342, Springer, Heidelberg, Germany, 60-88, 2006.
[20] L.S. Shapley and M. Shubik. A method for evaluating the distribution of power in a committee system. *American Political Science Review* 48: 787-792, 1954.